EQUIVARIANT UNIRATIONALITY OF TORI IN SMALL DIMENSIONS

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ABSTRACT. We study equivariant unirationality of actions of finite groups on tori of small dimensions.

1. Introduction

Rationality of tori over nonclosed fields is a well-established and active area of research, going back to the work of Serre, Voskresenskiĭ, Endo, Miyata, Colliot-Thélène, Sansuc, Saltman, Kunyavskiĭ (classification of rational tori in dimension 3), and to more recent contributions of Lemire and Hoshi–Yamasaki (stably rational classification in dimensions ≤ 5), see, e.g., [13] for a summary of results and extensive background material.

In pursuing analogies between birational geometry over nonclosed fields and equivariant birational geometry, i.e., birational geometry over the classifying stack BG, where G is a finite group, it is natural to consider algebraic tori in both contexts. While some of the invariants have a formally similar flavor, e.g., invariants of the geometric character lattice as a Galois, respectively, G-module, there are also striking differences. For example, a major open problem is to find examples of stably rational but nonrational tori over nonclosed fields. Over BG, there are examples already in dimension 2 [18, Section 9]. Furthermore, "rational" tori over BG need not have G-fixed points!

To make this dictionary more precise: in the equivariant setup, one studies regular, but not necessarily generically free, actions of finite groups G on smooth projective rational varieties X, over an algebraically closed field of characteristic zero. The following properties, the equivariant analogs of the notions of *(stable) rationality* and *unirationality*, have attracted attention:

- **(L), (SL)** *lineariazability*, respectively, *stable linearizability*: there exists a linear representation V of G and a G-equivariant birational map

$$\mathbb{P}(V) \dashrightarrow X, \quad \text{respectively,} \quad \mathbb{P}(V) \dashrightarrow X \times \mathbb{P}^m,$$

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with trivial action on the $\mathbb{P}^m$-factor,
- **(U)** *unirationality*: there exists a linear representation V of G and a G-equivariant dominant rational map

$$\mathbb{P}(V) \dashrightarrow X.$$

Property **(U)** is also known as *very versality* of the G-action. It was explored in the context of essential dimension in, e.g., [9]. It has been studied for del Pezzo surfaces in [8], and for toric varieties in [7, 15].

A necessary condition for both **(SL)** and **(U)** is
- **(A)**: for every abelian subgroup $A \subseteq G$ one has $X^A \neq \emptyset$.

A necessary condition for **(SL)** is:
- **(SP)**: the Picard group $\mathrm{Pic}(X)$ is a stably permutation G-module.

A necessary condition for **(U)** is:
- **(T)**: the action lifts to the universal torsor, see [15, Section 5] and Section 2 for more details.

These conditions are equivariant stable birational invariants of smooth projective varieties. We have

$$(\mathbf{SL}) \quad \Rightarrow \quad (\mathbf{U}),$$

but the converse fails already for del Pezzo surfaces: there exist quartic del Pezzo surfaces satisfying **(U)** but failing **(SP)**, and thus **(SL)**. By [8, Theorem 1.4], Condition **(A)** is sufficient for **(U)**, for regular, generically free actions on del Pezzo surfaces of degree ≥ 3; same holds for smooth quadric threefolds, or intersections of two quadrics in $\mathbb{P}^5$, by [5]. In [15] it was shown that regular, not necessarily generically free, actions on toric varieties are unirational, if and only if **(T)** is satisfied. Using this, and [12, Proposition 12], we have, for G-actions on toric varieties arising from an injective homomorphism $G \hookrightarrow \mathrm{Aut}(T)$,

$$(\mathbf{U}) + (\mathbf{SP}) \Longleftrightarrow (\mathbf{SL}).$$

Our goal in this note is to obtain explicit, group-theoretic, criteria for stable linearizability and unirationality of actions of finite groups G on smooth projective equivariant compactifications of tori $T = \mathbb{G}_m^n$ in small dimensions. When $n = 2$, Condition **(A)** implies **(U)** and **(SL)** [8, 12]. Our main result is:

Theorem 1.1. *Let $T = \mathbb{G}_m^3$, $G \subset \mathrm{Aut}(T)$ be a finite group, and X a smooth projective G- and T-equivariant compactification of T. Let*

$$\pi^* : G \to \mathsf{GL}(\mathrm{N})$$

be the induced representation on the cocharacter lattice N *of* T. *Assume that the* G*-action on* X *satisfies Condition* **(A)**. *Then*

- *the* G*-action is* **(U)** *if and only if* $\pi^*(G)$ *does not contain, up to conjugation, the group*
$$\left\langle \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \right\rangle \simeq C_2^2, \tag{1.1}$$
- *the* G*-action is* **(SL)** *if and only if* $\pi^*(G)$ *does not contain, up to conjugation, the group in* (1.1) *or any of the groups*
$$\left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle \simeq C_2 \times C_4,$$
$$\left\langle \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle \simeq C_2^3.$$

In particular,
$$(\mathbf{A}) + (\mathbf{SP}) \iff (\mathbf{SL}).$$

Here is the roadmap of the paper: In Section 2 we recall basic toric geometry and group cohomology. In Section 3 we provide details on equivariant geometry of toric surfaces. In Section 4 we recall the construction of equivariant smooth projective models of 3-dimensional tori, following [17]. In Section 5 we prove the main technical lemmas needed for Theorem 1.1. A particularly difficult case, with $\pi^*(G)$ given by (1.1), is outsourced to Section 6. In Section 7 we summarize the main steps of the proof of Theorem 1.1.

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2. Generalities

We work over an algebraically closed field k of characteristic zero.

Automorphisms. Let $T = \mathbb{G}_m^n$ be an algebraic torus over k. The automorphisms of T admit a description via the exact sequence
$$1 \to T(k) \to \mathrm{Aut}(T) \xrightarrow{\pi} \mathsf{GL}(\mathrm{M}) \to 1,$$
where $\mathrm{M} := \mathfrak{X}^*(T)$ is the character lattice of T, which is dual to the cocharacter lattice N. For any finite subgroup $G \subset \mathrm{Aut}(T)$, we have an exact sequence
$$1 \to G_T \to G \stackrel{\pi}{\to} \bar{G} \to 1, \quad G_T := T(k) \cap G. \tag{2.1}$$
The cases where G fixes a point on T, without loss of generality $1 \in T$, were studied in the context of geometry over nonclosed fields in [17, 13].

By way of contrast, here we allow the more general actions considered in [12], where X is an equivariant compactification of a *torsor* under a G-torus, these can still be (stably) linearizable.

As a convention, the G-action is from the right throughout the paper. For example, choosing appropriate coordinates $\{t_1, t_2, t_3\}$ on $T = \mathbb{G}_m^3$, the matrices in (1.1) correspond to actions on T given by

$$(t_1, t_2, t_3) \mapsto (t_2, t_1, \frac{1}{t_1 t_2 t_3}), \quad (t_1, t_2, t_3) \mapsto (\frac{1}{t_1 t_2 t_3}, t_3, t_2).$$

Smooth projective models. To obtain a smooth, projective, G- and T-equivariant compactification X of T, it suffices to choose a $\pi^*(G)$-invariant complete regular *fan* Σ in the lattice of cocharacters N, see, e.g., [2, Section 1.3]. Indeed, the translation action by $T(k)$ extends to any such compactification of T, by definition. Such a choice of X and Σ yields two exact sequences of G-modules

$$1 \to k^\times \to k(T)^\times \to \mathrm{M} \to 0, \tag{2.2}$$

and

$$0 \to \mathrm{M} \to \mathrm{PL} \to \mathrm{Pic}(X) \to 0, \tag{2.3}$$

where PL is a free $\mathbb{Z}$-module with generators corresponding to 1-dimensional cones in Σ, or equivalently, irreducible components of $X \setminus T$.

Obstruction class. The Yoneda product of the extensions (2.2) and (2.3) yields a cohomology class

$$\beta(X, G) \in \mathrm{Ext}^2(\mathrm{Pic}(X), k^\times) \simeq \mathrm{H}^2(G, \mathrm{Pic}^\vee \otimes k^\times) \simeq \mathrm{H}^3(G, \mathrm{Pic}(X)^\vee).$$

Effectively, $\beta(X, G)$ can be computed as the image of

$$\mathrm{id}_{\mathrm{Pic}(X)} \in \mathrm{End}(\mathrm{Pic}(X))^G$$

under the composition of the following connecting homomorphisms, arising from tensoring (2.2) and (2.3) by $\mathrm{Pic}(X)^\vee$:

$$\mathrm{End}(\mathrm{Pic}(X))^G \to \mathrm{H}^1(G, \mathrm{Pic}(X)^\vee \otimes \mathrm{M}) \to \mathrm{H}^2(G, \mathrm{Pic}(X)^\vee \otimes k^\times). \tag{2.4}$$

As explained in [15, Section 5], if $Y \to X$ is a G-equivariant morphism then

$$\beta(Y, G) = 0 \quad \Rightarrow \quad \beta(X, G) = 0. \tag{2.5}$$

For toric varieties, this is also a consequence of Theorem 2.4 below. By basic properties of cohomology, we observe:

Lemma 2.1. *We have*

$$\beta(X, G) = 0 \iff \beta(X, G_p) = 0 \ \textit{ for all } p\textit{-Sylow subgroups } G_p \subseteq G.$$

Proof. Since G is finite, for any G-module P, the sum of restriction homomorphisms gives an embedding

$$\mathrm{H}^2(G,\mathrm{P}) \to \oplus_p \mathrm{H}^2(G_p,\mathrm{P}),$$

where p runs over primes dividing $|G|$. $\square$

Bogomolov multiplier. As explained in [12, Section 3.6], functoriality implies that

$$X^G \neq \emptyset \quad \Rightarrow \quad \beta(X,G) = 0. \tag{2.6}$$

Thus Condition **(A)** forces that

$$\beta(X,G) \in \mathrm{B}^3(G,\mathrm{Pic}(X)^\vee), \tag{2.7}$$

where for any G-module P and $n \in \mathbb{N}$, we put

$$\mathrm{B}^n(G,\mathrm{P}) := \bigcap_A \mathrm{Ker}\left(\mathrm{H}^n(G,\mathrm{P}) \xrightarrow{\mathrm{res}} \mathrm{H}^n(A,\mathrm{P})\right),$$

the intersection over all abelian subgroups $A \subseteq G$; this is the generalization of the Bogomolov multiplier

$$\mathrm{B}^2(G,k^\times),$$

where the G-action on $k^\times$ is trivial, considered in [20, Section 2]. Note that while the vanishing of $\beta(X,G)$ is a stable birational invariant, the group $\mathrm{B}^n(G,\mathrm{Pic}(X)^\vee)$ is not, in general, as the following example shows.

Example 2.2. Let G be a group with a nontrivial Bogomolov multiplier. Let V be a faithful linear representation of G and $X = \mathbb{P}(\mathbf{1} \oplus V)$. Let $\tilde{X}$ be the blowup of X in the G-fixed point. Then

$$\mathrm{B}^3(G,\mathrm{Pic}(\tilde{X})^\vee) \neq \mathrm{B}^3(G,\mathrm{Pic}(X)^\vee) \oplus \mathrm{B}^3(G,\mathbb{Z}).$$

We will need the following technical statement about generalized Bogomolov multipliers:

Lemma 2.3. *Let $H \subset G$ be a normal subgroup with cyclic quotient $G/H = C_m$. Let P_0 be an H-module and $\mathrm{P} = \oplus_{j=1}^m \mathrm{P}_0$ the induced G-module. Then the restriction homomorphism*

$$\mathrm{H}^2(G,\mathrm{P}) \to \mathrm{H}^2(H,\mathrm{P})$$

is injective. In particular, we have

$$\mathrm{B}^2(H,\mathrm{P}_0) = 0 \quad \Rightarrow \quad \mathrm{B}^2(G,\mathrm{P}) = 0.$$

Proof. The Hochschild–Serre spectral sequence yields:

$$\begin{aligned} 0 \to \mathrm{H}^1(G/H,\mathrm{P}^H) \to \mathrm{H}^1(G,\mathrm{P}) \to \mathrm{H}^1(H,\mathrm{P})^{G/H} \to \mathrm{H}^2(G/H,\mathrm{P}^H) \to \\ \to \ker\left(\mathrm{H}^2(G,\mathrm{P}) \to \mathrm{H}^2(H,\mathrm{P})\right) \to \mathrm{H}^1(G/H,\mathrm{H}^1(H,\mathrm{P})) \end{aligned}$$

We have

$$\mathrm{H}^2(G/H, \mathrm{P}^H) = \mathrm{H}^1(G/H, \mathrm{H}^1(H, \mathrm{P})) = 0.$$

Indeed, G/H-acts via cyclic permutations on the summands of P^H and $\mathrm{H}^1(H, \mathrm{P})$; cohomology of cyclic groups acting via cyclic permutations vanishes in all degrees ≥ 1. It follows that

$$\ker\left(\mathrm{H}^2(G, \mathrm{P}) \to \mathrm{H}^2(H, \mathrm{P})\right) = 0.$$

Thus, a nonzero class $\alpha \in \mathrm{H}^2(G, \mathrm{P})$ remains nonzero in

$$\mathrm{H}^2(H, \mathrm{P}) = \oplus_{j=1}^m \mathrm{H}^2(H, \mathrm{P}_0).$$

If $\alpha \in \mathrm{B}^2(G, \mathrm{P})$, then the restriction of α to H lies in $\mathrm{B}^2(H, \mathrm{P})$, contradicting the assumption that $\mathrm{B}^2(H, \mathrm{P}_0) = 0$. $\square$

Geometric applications. The following theorem characterizes unirationality of G-actions on toric varieties:

Theorem 2.4. [12, Section 4], [15, Section 5] *Let X be a smooth projective T-equivariant compactification of a torus T, with an action of a finite group G arising from a homomorphism $\rho : G \to \mathrm{Aut}(T)$. Then*

$$\beta(X, G) = 0 \iff (\mathbf{T}) \iff (\mathbf{U}).$$

A related result, concerning *versality* of generically free actions on toric varieties is [7, Theorem 3.2]: it is equivalent to the vanishing of $\beta(X, G)$, in our terminology. In particular, for such actions, versality is equivalent to *very versality* (i.e., unirationality), see [9] for further details regarding these notions. Here, we allow actions with nontrivial generic stabilizers, i.e., when the kernel of ρ is nontrivial.

A consequence of [12, Proposition 12] is:

Theorem 2.5. *Let X be a smooth projective T-equivariant compactification of a torus T, with a regular action of a finite group G arising from an injective homomorphism $\rho : G \hookrightarrow \mathrm{Aut}(T)$. Then*

$$(\mathbf{T}) + (\mathbf{SP}) \iff (\mathbf{SL}).$$

Note that

$$(\mathbf{A}) \not\Rightarrow (\mathbf{U}),$$

even for generically free actions on toric threefolds, see Section 4. In fact, not even for $X = \mathbb{P}^1$, if we allow generic stabilizers!

Example 2.6. Consider $X = \mathbb{P}^1$. If $G \subset \mathrm{Aut}(T)$, i.e.,

$$G \subset \mathbb{G}_m(k) \rtimes C_2,$$

then G is either cyclic or dihedral. Actions of cyclic groups and of dihedral groups $\mathfrak{D}_n$ of order $2n$, with n odd, are unirational; actions of

$\mathfrak{D}_n$, with n even, are not unirational, since they contain the subgroup C_2^2 which has no fixed points on $\mathbb{P}^1$, i.e., failing Condition **(A)**. In particular, for generically free actions we have (**A**) $\Leftrightarrow$ (**U**).

For non-generically free G-actions on $\mathbb{P}^1$, considered in [15, Example 2.2], Condition **(A)** does not suffice to characterize unirationality. By (2.7), such groups must have a nontrivial Bogomolov multiplier

$$\mathrm{B}^2(G, k^\times) \simeq \mathrm{B}^3(G, \mathbb{Z}).$$

For example, let G be the group of order 64, with `GAP ID (64,149)`; this is the smallest group with nontrivial $\mathrm{B}^2(G, k^\times)$. There is a unique subgroup $H \simeq C_2 \times \mathfrak{Q}_8 \subset G$, with `GAP ID (16,12)`, and $G/H \simeq C_2^2$. Consider a homomorphism $\rho : G \to \mathsf{PGL}_2(k)$ with kernel H and image C_2^2. The resulting G-action on $\mathbb{P}^1$ is not generically free, but satisfies Condition **(A)** – no abelian subgroup of G surjects onto C_2^2, via ρ. We compute as in (2.4) that

$$0 \neq \beta(\mathbb{P}^1, G) \in \mathrm{B}^2(G, k^\times) \subset \mathrm{H}^2(G, k^\times) = \mathrm{H}^2(G, \mathrm{Pic}(\mathbb{P}^1)^\vee \otimes k^\times).$$

Another way to see this is to observe the equality of commutator subgroups

$$[G, G] = [G, H],$$

which is impossible if the extension of G by $k^\times$ associated with the class $\beta(\mathbb{P}^1, G)$ splits, see [15, Example 2.2]. Thus, **(U)** fails for this action.

Group cohomology. The computation of the obstruction class $\beta(X, G)$ relies on an explicit resolution of the group ring. We write down such resolutions for groups that will be relevant for the analysis of 3-dimensional toric varieties in Section 4. By convention, we work with right G-modules, i.e., the group G acts from the right.

- Let $G = \mathfrak{Q}_{2^n}$ be the generalized quaternion group of order 2^n, with a presentation

$$\mathfrak{Q}_{2^n} := \langle x, y | x^{2^{n-2}} = y^2,\ xyx = y\rangle,$$

and P a G-module. By [3, §XII.7], the cohomology groups $\mathrm{H}^i(G, \mathrm{P})$, $i = 0, 1, 2, 3$, can be computed as the i-th cohomology of the complex

$$\mathrm{P} \xrightarrow{\begin{pmatrix} 1-x & 1-y \end{pmatrix}} \mathrm{P}^2 \xrightarrow{\begin{pmatrix} N_x & yx+1 \\ -1-y & x-1 \end{pmatrix}} \mathrm{P}^2 \xrightarrow{\begin{pmatrix} 1-x \\ yx-1 \end{pmatrix}} \mathrm{P} \xrightarrow{\sum_{g\in G} g} \mathrm{P} \cdots \qquad (2.8)$$

where

$$N_x = 1 + x + x^2 + \cdots + x^{(2^{n-2}-1)}.$$

- Let $G = \mathfrak{D}_{2^{n-1}}$ be the dihedral group of order 2^n, with a presentation

$$\mathfrak{D}_{2^{n-1}} := \langle x, y | x^{2^{n-1}} = y^2 = yxyx = 1\rangle,$$

and P a G-module. By [1, §IV.2], the cohomology groups $\mathrm{H}^i(\mathfrak{D}_n, \mathrm{P})$, $i = 0, 1, 2$, can be computed as the i-th cohomology of

$$\mathrm{P} \xrightarrow{\begin{pmatrix} 1-x & 1-y \end{pmatrix}} \mathrm{P}^2 \xrightarrow{\begin{pmatrix} N_x & 1+yx & 0 \\ 0 & x-1 & 1+y \end{pmatrix}} \mathrm{P}^3 \xrightarrow{\begin{pmatrix} 1-x & 1+y & 0 & 0 \\ 0 & -N_x & 1-yx & 0 \\ 0 & 0 & 1-x & 1-y \end{pmatrix}} \mathrm{P}^4 \cdots \tag{2.9}$$

where

$$N_x = 1 + x + x^2 + \cdots + x^{(2^{n-1}-1)}.$$

• Let $G = \mathfrak{SD}_{2^n}$ be the semidihedral group of order 2^n, with presentation

$$\mathfrak{SD}_{2^n} := \langle x, y | x^{2^{n-1}} = y^2 = 1,\ yxy = x^{2^{n-2}-1} \rangle,$$

and P a G-module. Using the resolution constructed in [10], we can compute $\mathrm{H}^i(G, \mathrm{P})$, $i = 0, 1, 2$, as the i-th cohomology of

$$\mathrm{P} \xrightarrow{\begin{pmatrix} 1-x & 1-y \end{pmatrix}} \mathrm{P}^2 \xrightarrow{\begin{pmatrix} L_2 & 0 \\ L_1 & 1+y \end{pmatrix}} \mathrm{P}^2 \xrightarrow{\begin{pmatrix} -L_3 & 0 \\ L_4 & 1-y \end{pmatrix}} \mathrm{P}^2 \cdots \tag{2.10}$$

where

$$L_1 = x^{2^{n-3}+1} - 1, \quad L_2 = \sum_{r=0}^{2^{n-3}} x^r - \left(\sum_{r=0}^{2^{n-3}-2} x^r \right) \cdot y,$$
$$L_3 = (x^{2^{n-3}-1} - 1)(1+y), \quad L_4 = (x^{2^{n-3}+1} - 1)(x^{2^{n-3}-1} - 1).$$

3. Toric surfaces

In this section, we recall the classification of unirational and linearizable actions of subgroups $G \subset \mathrm{Aut}(T)$ on smooth projective toric surfaces $X \supset T$. The maximal finite subgroups of $\mathsf{GL}_2(\mathbb{Z})$ are

$$\mathfrak{D}_4 \quad \text{and} \quad \mathfrak{D}_6.$$

The recipe of Section 2 shows that all actions as above can be realized as regular actions on $X = \mathbb{P}^1 \times \mathbb{P}^1$ and respectively, $X = \mathrm{dP}_6$, the del Pezzo surface of degree 6.

Proposition 3.1. *Let X be a smooth projective toric surface with an action of a finite group $G \subset \mathrm{Aut}(T)$. Then*

$$(\mathbf{SL}) \iff (\mathbf{U}) \iff (\mathbf{A}).$$

Proof. The right equivalence is proved in [8]. The left equivalence follows from the general result Theorem 2.5, i.e., [12, Proposition 12]: if the generically free G-action satisfies **(T)** and $\mathrm{Pic}(X)$ is a stably permutation G-module, then the action is stably linearizable. The stable permutation property of the G-action on $\mathrm{Pic}(X)$ is clear for $X = \mathbb{P}^1 \times \mathbb{P}^1$; for $X = \mathrm{dP6}$, see [12, Section 6]. $\square$

It will be convenient to choose coordinates $\{t_1, t_2\}$ of $T = \mathbb{G}_m^2$. This determines a basis $\{m_1, m_2\}$ of the lattice M and $\{n_1, n_2\}$ of its dual N. Assume that in the basis $\{n_1, n_2\}$, the $\mathfrak{D}_4$-action is generated by the involutions

$$\iota_1 := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \iota_2 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \iota_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consider the involutions in $T(k) \cap G$, in the coordinates $\{t_1, t_2\}$,

$$\tau_1 := (-1, -1), \quad \tau_2 := (-1, 1), \quad \tau_3 := (1, -1).$$

A classification of G yielding versal (and thus unirational) actions can be found in [7, Section 4.1]. Note that only 2 and 3-groups are relevant.

Proposition 3.2. *Let $G \subset \mathrm{Aut}(T)$ be a finite subgroup acting on a smooth projective toric surface $X \supset T$. Then*

- *if G is a 3-group, then the G-action is unirational if and only if $\pi^*(G) = 1$ or $G_T = 1$.*
- *if G is a 2-group, then the G-action is unirational if and only if one of the following holds, up to conjugation,*
  - $\pi^*(G) = 1$ *or* $\langle \iota_3 \rangle$,
  - $\pi^*(G) = \langle \iota_2 \rangle$ *and* $G_T \subset \{(1, t) : t \in \mathbb{G}_m(k)\} \subset T(k)$,
  - $G_T = 1$ *otherwise.*

Moreover, for actions of p-groups, unirationality implies linearizability.

Here, we recall the classification on linearizable G-actions on toric surfaces, with $G \subset \mathrm{Aut}(T)$, following [6] and [19]. Consider the case $\pi^*(G) \subseteq \mathfrak{D}_4$:

- If $\mathrm{rk}\,\mathrm{Pic}(X)^G = 1$ then the G-action is linearizable if and only if $\pi^*(G)$ is conjugate to $\langle \iota_3 \rangle \simeq C_2$ in $\mathsf{GL}_3(\mathbb{Z})$.
- If $\mathrm{rk}\,\mathrm{Pic}(X)^G = 2$ then the G-action is linearizable if and only if up to conjugation one of the following holds:
  - $\pi^*(G) = \langle \iota_2 \rangle$ and
  $$G_T \subset \{(t_1, t_2) : t_1, t_2 \in \mathbb{G}_m(k), \quad \mathrm{ord}(t_1) \text{ is odd}\}.$$
  - $\pi^*(G) = \langle \iota_1 \rangle$ or $\langle \iota_1, \iota_2 \rangle$ and $|G_T|$ is odd.

We turn to $\pi^*(G) \subseteq \mathfrak{D}_6$:

- If $\mathrm{rk}\,\mathrm{Pic}(X)^G = 1$ then the G-action is linearizable if and only if $G_T = 1$ and $G \simeq C_6$ or $G \simeq \mathfrak{S}_3$.
- If $\mathrm{rk}\,\mathrm{Pic}(X)^G = 2$ and $\pi^*(G) = C_3$ or $\mathfrak{S}_3$, then the action is linearizable if and only if $3 \nmid |G_T|$; all other possibilities for $\pi^*(G)$ are realized as subgroups of $\mathfrak{D}_4$, covered above.

4. Toric threefolds: smooth projective models

We start with the classification of actions and their realizations on smooth projective toric threefolds, following [17]. Let $G \subset \mathrm{Aut}(T)$ be a finite subgroup where $T = \mathbb{G}_m^3$. Recall from the exact sequence (2.1) that $\bar{G} = \pi(G)$ is a subgroup of $\mathsf{GL}_3(\mathbb{Z})$. There are two isomorphism classes of maximal finite subgroups of $\mathsf{GL}_3(\mathbb{Z})$:

$$C_2 \times \mathfrak{S}_4 \quad \text{and} \quad C_2 \times \mathfrak{D}_6.$$

The first group gives *three* conjugacy classes in $\mathsf{GL}_3(\mathbb{Z})$, referred to as Case **(C)**, **(S)**, and **(P)**, respectively. The other group gives one conjugacy class, called Case **(F)**.

Case (C). Here, $X = (\mathbb{P}^1)^3$, with $\bar{G} \subset C_2 \times \mathfrak{S}_4$ and the action visible from the presentation

$$1 \to C_2^3 \to C_2 \times \mathfrak{S}_4 \to \mathfrak{S}_3 \to 1,$$

with $\mathfrak{S}_3$ permuting the factors and C_2 acting as an involution on the corresponding $\mathbb{P}^1$.

Case (F). In this case, $X = \mathbb{P}^1 \times \mathrm{dP6}$, and $\bar{G} \subset C_2 \times \mathfrak{D}_6$, with C_2 acting via the standard involution on $\mathbb{P}^1$, and $\mathfrak{D}_6$ acting on dP6 as described in Section 3.

Case (P). In this case, X is the blowup of

$$\{u_1u_2u_3u_4 = v_1v_2v_3v_4\} \subset \mathbb{P}^1_{u_1,v_1} \times \mathbb{P}^1_{u_2,v_2} \times \mathbb{P}^1_{u_3,v_3} \times \mathbb{P}^1_{u_4,v_4}$$

in its 6 singular points, and $\bar{G} \subset C_2 \times \mathfrak{S}_4$. The corresponding $\pi^*(G)$-invariant fan Σ consists of 99 cones: 32 three-dimensional cones, 48 two-dimensional cones, 18 rays, and the origin. We have

$$\mathrm{Pic}(X) = \mathbb{Z}^{15}.$$

Case (S). Here, X is the blowup of $\mathbb{P}^3$ in 4 points and the 6 lines through these points, with $\bar{G} = C_2 \times \mathfrak{S}_4$, acting via permutations on the 4 points and 6 lines, with C_2 corresponding to the Cremona involution on $\mathbb{P}^3$, which is regular on X. A singular model is the intersection of two quadrics

$$\{y_1y_4 - y_2y_5 = y_1y_4 - y_3y_6\} \subset \mathbb{P}^5_{y_1,y_2,y_3,y_4,y_5}.$$

Blowing up its 6 singular points one obtains X, see [11, Section 9] for an extensive discussion of this geometry. The corresponding fan Σ consists of 75 cones: 24 three-dimensional cones, 36 two-dimensional cones, 14 rays, and the origin. We have

$$\mathrm{Pic}(X) = \mathbb{Z}^{11}.$$

By Lemma 2.1, to establish property **(U)** it suffices to consider p-Sylow subgroups of G, in our case, $p = 3$ or 2.

Models for 3-groups. The are two finite 3-subgroups of $\mathsf{GL}_3(\mathbb{Z})$, both isomorphic to C_3. They are generated by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{respectively,} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The corresponding model X can be chosen to be $\mathbb{P}^3$ and $\mathbb{P}^1 \times \mathbb{P}^2$ respectively.

Models for 2-groups. There are three conjugacy classes of $C_2 \times \mathfrak{S}_4$ in $\mathsf{GL}_3(\mathbb{Z})$, but only *two* conjugacy classes of their 2-Sylow subgroups, generated respectively by

$$\left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \simeq C_2 \times \mathfrak{D}_4,$$

and

$$\left\langle \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix} \right\rangle \simeq C_2 \times \mathfrak{D}_4.$$

The first group is realized on $X = (\mathbb{P}^1)^3$, and the other on either **(S)** or **(P)** model.

In the analysis below, we need a simpler smooth projective model when $\pi^*(G)$ is contained in the $\mathfrak{D}_4 \subset \mathsf{GL}_3(\mathbb{Z})$ generated by

$$\tau_1 := \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_2 := \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Let Σ be the fan in N generated by 6 rays with generators

$$\begin{array}{lll} v_1 = (-1, 0, -1), & v_2 = (0, -1, 0), & v_3 = (0, 0, 1), \\ v_4 = (1, 0, 0), & v_5 = (1, 1, 1), & v_6 = (1, 0, 1). \end{array}$$

and 8 cones

$$S_1 = \langle v_1, v_4, v_5 \rangle, S_2 = \langle v_1, v_3, v_5 \rangle, S_3 = \langle v_1, v_2, v_3 \rangle, S_4 = \langle v_1, v_2, v_4 \rangle,$$

$$S_5 = \langle v_4, v_5, v_6 \rangle, S_6 = \langle v_3, v_5, v_6 \rangle, S_7 = \langle v_2, v_3, v_6 \rangle, S_8 = \langle v_2, v_4, v_6 \rangle.$$

Then Σ is $\pi^*(G)$-invariant and the toric variety $X = X(\Sigma)$ is the blowup of a cone over a smooth quadric surface at its vertex.

5. Toric threefolds: unirationality

Let $T=\mathbb{G}_m^3$ and $G\subset \mathrm{Aut}(T)$ be a finite group, acting on a smooth projective X, which is a G- and T-equivariant compactification of T. We recall the exact sequence

$$1\to G_T\to G\xrightarrow{\pi}\bar{G}\to 1.$$

In this section, we classify unirational G-actions, in particular, these satisfy Condition **(A)**.

Proposition 5.1. *Let $G\subset \mathrm{Aut}(T)$ be 3-group such that the G-action on X satisfies Condition* **(A)**. *Then it satisfies* **(U)**.

Proof. As explained above, the model X can be chosen to be either $\mathbb{P}^3$ or $\mathbb{P}^1\times\mathbb{P}^2$. In the first case, the action is linear. When $X=\mathbb{P}^1\times\mathbb{P}^2$, the G-action on $\mathrm{Pic}(X)$ is trivial and

$$\beta(X,G)\in \mathrm{H}^2(G,k^\times)\oplus \mathrm{H}^2(G,k^\times);$$

by [15, Remark 5.5], obstruction to $\beta(X,G)=0$ equals the Amitsur obstruction for each factor. We have an extension

$$1\to G_T\to G\to C_3\to 1,$$

with G_T abelian; the Bogomolov multiplier $\mathrm{B}^2(G,k^\times)=0$, by, e.g., [16, Lemma 3.1]. Condition **(A)** implies that the G-action lifts to a linear action on each factor. □

Proposition 5.2. *Let $G\subset \mathrm{Aut}(T)$ be 2-group such that the G-action on X satisfies Condition* **(A)**. *Then it satisfies* **(U)** *if and only if one of the following holds:*

- $G_T=1$, *or*
- $G_T\neq 1$ *and $\pi^*(G)$ is not conjugated to*

$$\mathfrak{K}_9=\left\langle \begin{pmatrix}0&1&-1\\1&0&-1\\0&0&-1\end{pmatrix},\begin{pmatrix}-1&0&0\\-1&0&1\\-1&1&0\end{pmatrix}\right\rangle \simeq C_2^2. \tag{5.1}$$

Proof. The assertion follows from Lemmas 5.3, 5.4, 5.5, 5.7, 5.8, and 5.10. □

The rest of this section is devoted to the proof of this proposition. There are 2 conjugacy classes of maximal 2-groups in $\mathsf{GL}_3(\mathbb{Z})$, both isomorphic to

$$C_2\times\mathfrak{D}_4.$$

The corresponding toric models are **(C)**, and **(S)** or **(P)**. We proceed with a case-by-case analysis of actions; altogether, we have to consider 36 conjugacy classes of finite subgroups $\pi^*(G)\subset\mathsf{GL}(\mathrm{N})$. We summarize:

- When $G_T = 1$: **(U)** holds, by Lemma 5.3.
- When $G_T \neq 1$ and $\pi^*(G)$ isomorphic to
  - C_2, C_4: **(U)** holds, by Lemmas 5.5 and 5.7.
  - C_2^2: see Lemma 5.8.
  - $\mathfrak{D}_4$: see Lemma 5.9.
  - C_2^3, $C_2 \times C_4$, $C_2 \times \mathfrak{D}_4$: all such actions fail Condition **(A)**, by Lemma 5.4.

Lemma 5.3. *Assume that $G \subset \mathrm{Aut}(T)$ is a 2-group with $G_T = 1$. Then*

$$\mathbf{(A)} \iff \mathbf{(U)}.$$

Proof. When G is abelian, the claim follows from (2.6). It remains to consider the cases when $G = \mathfrak{D}_4$ or $C_2 \times \mathfrak{D}_4$. Via `HAP`, we have computed that for corresponding models X, i.e., **(C)** or **(P)**, the generalized Bogomolov multiplier satisfies

$$\mathrm{B}^2(G, \mathrm{Pic}(X)^\vee \otimes k^\times) = 0.$$

Condition **(A)** implies that

$$\beta(X, G) \in \mathrm{B}^2(G, \mathrm{Pic}(X)^\vee \otimes k^\times),$$

and thus $\beta(X, G) = 0$; it remains to apply Theorem 2.4. □

Note that X may fail to have G-fixed points even when $G_T = 1$, see [4, Remark 5.2].

From now on, we assume that

- $G \subset \mathrm{Aut}(T)$ is a 2-group and
- $G_T \neq 1$.

Lemma 5.4. *Assume that $\pi^*(G)$ contains*

$$\eta := \mathrm{diag}(-1, -1, -1) \in \mathsf{GL}(\mathrm{N}).$$

Then the G-action fails Condition **(A)**.

Proof. Indeed, η can be realized as the diagonal involution on $(\mathbb{P}^1)^3$, and any translation by a 2-group will produce a C_2^2 action without fixed points. □

After excluding groups containing η, it suffices to consider

$$\pi^*(G) = C_2, \quad C_4, \quad C_2^2, \quad \mathfrak{D}_4.$$

Lemma 5.5. *Assume that $\pi^*(G) = C_2$. Then*

$$\mathbf{(A)} \iff \mathbf{(U)}.$$

Proof. Apart from $\langle\eta\rangle$, there are 4 conjugacy classes of groups of order 2 in $\mathsf{GL}_3(\mathbb{Z})$, generated by

$$\iota_1=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\ \iota_2=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix},\ \iota_3=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\ \iota_4=\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}.$$

The first case is realized on $(\mathbb{P}^1)^3$ and the last three cases in $\mathbb{P}^2\times\mathbb{P}^1$. By [15, Remark 5.2], unirationality is determined by unirationality of all of the $\mathbb{P}^1$ and $\mathbb{P}^2$ factors, which is equivalent to triviality of the Amitsur invariant, see Example 2.6.

Since G is an extension of the cyclic group C_2 by an abelian group G_T, the Bogomolov multiplier $\mathrm{B}^2(G,k^\times)=0$. Together with Condition **(A)** this implies that the Amitsur invariant for the action on each factor is trivial, and the G-action on X is unirational. □

Remark 5.6. Alternatively, one can check that for $\pi^*(G)=C_2$, the G-action satisfies Condition **(A)** if and only if one of the following holds:

- $\pi^*(G)=\langle\iota_1\rangle$, and $G_T\subset\{(t,1,1):t\in\mathbb{G}_m(k)\}\subset T(k)$.
- $\pi^*(G)=\langle\iota_2\rangle$, and $G_T\subset\{(t_1,t_2,1):t_1,t_2\in\mathbb{G}_m(k)\}\subset T(k)$.
- $\pi^*(G)=\langle\iota_3\rangle$, and G_T is any subgroup of $T(k)$.
- $\pi^*(G)=\langle\iota_4\rangle$, and $(t,t,-1)\notin G_T$ for any $t\in\mathbb{G}_m(k)$.

Using this description, we see that Condition **(A)** is also equivalent to $X^G\neq\emptyset$, in the first three cases.

Lemma 5.7. *Assume that* $\pi^*(G)=C_4$. *Then*

$$(\mathbf{A})\iff(\mathbf{U}).$$

Proof. There are 4 conjugacy classes of $C_4\subset\mathsf{GL}_3(\mathbb{Z})$, generated by

$$\theta_1=\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix},\theta_2=\begin{pmatrix}-1&0&0\\0&0&1\\0&-1&0\end{pmatrix},\theta_3=\begin{pmatrix}-1&-1&-1\\1&0&0\\0&1&0\end{pmatrix},\theta_4=\begin{pmatrix}1&1&1\\-1&0&0\\0&-1&0\end{pmatrix}.$$

The first two cases are realized on $\mathbb{P}^1\times Q$, where $Q=\mathbb{P}^1\times\mathbb{P}^1$. The third case on $\mathbb{P}^3$. The fourth can be realized on either the **(P)** or **(S)** model.

Case θ_1: Note that $\theta_1^2=\iota_1$. Condition **(A)** implies that

$$G_T\subset\{(t,1,1):t\in\mathbb{G}_m(k)\}\subset T(k)$$

and G fixes a point on $X=\mathbb{P}^1\times Q$; therefore, the G-action satisfies **(U)**.

Case θ_2: We also have $\theta_2^2=\iota_1$. Condition **(A)** implies that G_T contains

$$\iota=(-1,1,1)\in T(k).$$

However, for $g\in G$ such that $\pi^*(g)=\theta_2$, the abelian subgroup $\langle g,\iota\rangle$ of G does not fix points on X, contradiction.

Case θ_3: Let $g \in G$ be such that $\pi^*(g) = \theta_3$. Up to conjugation by an element in $T(k)$, we may assume that g acts on $\mathbb{P}^3_{y_1,y_2,y_3,y_4}$ via

$$(y_1, y_2, y_3, y_4) \mapsto (y_4, y_1, y_2, y_3).$$

One can check that for any 2-torsion element $\iota \in T(k)$, the group $\langle g, \iota\rangle$ contains an abelian subgroup with no fixed point on $\mathbb{P}^3$, contradiction.

Case θ_4: This element is contained in a $\mathfrak{D}_4$, covered in Lemma 5.9. $\square$

Lemma 5.8. *Assume that* $\pi^*(G) = C_2^2$. *Then*

$$(\mathbf{A}) \Longleftrightarrow (\mathbf{U}),$$

unless $\pi^*(G)$ *is conjugate to the group indicated in* (5.1).

Proof. We have 9 conjugacy classes of C_2^2 in $\mathsf{GL}_3(\mathbb{Z})$ not containing η, denoted by

$$\mathfrak{K}_1, \dots, \mathfrak{K}_9.$$

We study their realizations:

Cases $\mathfrak{K}_1$ and $\mathfrak{K}_2$: Here, $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and

$$\begin{aligned}\mathfrak{K}_1 =&\langle \mathrm{diag}(-1, -1, 1), \mathrm{diag}(-1, 1, -1)\rangle,\\ \mathfrak{K}_2 =&\langle \mathrm{diag}(1, 1, -1), \mathrm{diag}(-1, 1, -1)\rangle.\end{aligned}$$

Using Remark 5.6, we see that Condition **(A)** fails if $\pi^*(G) = \mathfrak{K}_1$. When $\pi^*(G) = \mathfrak{K}_2$, Condition **(A)** implies that

$$G_T \subset \{(1, t, 1) : t \in \mathbb{G}_m(k)\} \subset T(k).$$

In this case, $X^G \neq \emptyset$ and (**U**) holds.

Cases $\mathfrak{K}_3$, $\mathfrak{K}_4$, and $\mathfrak{K}_5$: Here, $X = \mathbb{P}^1 \times Q$, with G switching the factors in $Q = \mathbb{P}^1 \times \mathbb{P}^1$. The groups are

$$\mathfrak{K}_3 = \left\langle \begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\right\rangle, \quad \mathfrak{K}_4 = \left\langle \begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\right\rangle,$$

$$\mathfrak{K}_5 = \left\langle \begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}0&0&1\\0&-1&0\\1&0&0\end{pmatrix}\right\rangle.$$

When $\pi^*(G) = \mathfrak{K}_3$, Condition **(A)** implies that

$$G_T \subset \{(t_1, 1, t_2) : t_1, t_2 \in \mathbb{G}_m(k)\} \subset T(k).$$

When $\pi^*(G) = \mathfrak{K}_4$,

$$G_T \subset \{(1, t, 1) : t \in \mathbb{G}_m(k)\} \subset T(k).$$

In both cases, $X^G \neq \emptyset$ and thus **(U)**. However, when $\pi^*(G) = \mathfrak{K}_5$, the subgroup generated by

$$(1,-1,1) \in G_T, \quad \text{and a lift to } G \text{ of } \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is an abelian group without fixed points.

Cases $\mathfrak{K}_6$ and $\mathfrak{K}_7$: Here, $X = \mathbb{P}^3$, and

$$\mathfrak{K}_6 = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle, \quad \mathfrak{K}_7 = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix} \right\rangle.$$

When $\pi^*(G) = \mathfrak{K}_6$, then, up to conjugation, the G-action on $\mathbb{P}^3_{y_1,y_2,y_3,y_4}$ is given by G_T and a lift of $\mathfrak{K}_6$ generated by

$$g_1 : (\mathbf{y}) \mapsto (y_3, ay_4, y_1, ay_2), \quad a \in \mathbb{G}_m(k),$$
$$g_2 : (\mathbf{y}) \mapsto (y_4, b_1y_3, b_1y_2, b_2y_1), \quad b_1, b_2 \in \mathbb{G}_m(k).$$

For $I \subset \{1,2,3,4\}$, let s_I be the diagonal matrix changing the signs of $y_i, i \in I$. The abelian groups

$$\langle g_1, s_{\{1,2\}} \rangle, \quad \langle g_2, s_{\{1,3\}} \rangle, \quad \langle g_1g_2, s_{\{1,4\}} \rangle$$

have no fixed points on $\mathbb{P}^3$. On the other hand, we have that

$$(g_1g_2)^2 = \mathrm{diag}(1,1,b_2,b_2).$$

If $b_2 \neq 1$, then $s_{\{1,2\}} \in G_T$. If $b_2 = 1$, then

$$(s_{\{i\}}g_1g_2)^2 = s_{\{1,2\}} \in G_T,$$

for any $i = 1,2,3,$ or 4 such that $s_{\{i\}} \in G_T$. Thus, in all cases, Condition **(A)** fails.

When $\pi^*(G) = \mathfrak{K}_7$, G is generated by G_T, and

$$g_3 : (\mathbf{y}) \mapsto (c_1y_1, c_2y_2, y_4, y_3), \quad c_1, c_2 \in \mathbb{G}_m(k),$$
$$g_4 : (\mathbf{y}) \mapsto (y_2, y_1, c_3y_3, c_4y_4), \quad c_3, c_4 \in \mathbb{G}_m(k).$$

Note that the abelian group generated by

$$\langle g_3g_4, \mathrm{diag}(1,-1,a_1,-a_1) \rangle,$$

does not fix any points on $\mathbb{P}^3$, for any $a_1 \in \mathbb{G}_m(k)$.

If $c_1 \neq \pm c_2$ and $c_3 \neq \pm c_4$, then g_3 and g_4 generate one of

$$\mathrm{diag}(1,-1,1,-1) \quad \text{and} \quad \mathrm{diag}(1,-1,-1,1),$$

and Condition **(A)** fails. Thus, we may assume that $c_1 = \pm c_2$ and all elements in G_T are of the form

$$\mathrm{diag}(1,\pm 1, a_3, a_4), \quad a_3, a_4 \in \mathbb{G}_m(k), \quad a_3 \neq -a_4.$$

If all elements in G_T are of the form $\mathrm{diag}(1,1,a_3,a_4)$, then $\langle G_T, g_4\rangle$ is an abelian subgroup of G of index 2. It follows that the Bogomolov multiplier $\mathrm{B}^2(G,k^\times)=0$ and Condition **(A)** implies **(U)**.

Now, we consider the case when G_T contains elements of the form

$$\mathrm{diag}(1,-1,a_3,a_4).$$

Up to multiplying g_3 with such an element, we may assume that $c_1=-c_2$. We divide the argument into the following subcases:

(1) When $c_1=-c_2$, $c_3=c_4$ and all elements in G_T are of the form
$$\mathrm{diag}(1,\pm1,a_3,a_3),$$
then G fixes $[0:0:1:1]\in\mathbb{P}^3$.

(2) When $c_1=-c_2$ and $c_3=-c_4$, then $\langle g_3,g_4\rangle$ is an abelian group with no fixed points on $\mathbb{P}^3$.

(3) When $c_1=-c_2$, $c_3\neq\pm c_4$ and G_T contains an element
$$\varepsilon=\mathrm{diag}(1,\pm1,a_3,a_4),\quad a_3,a_4\in\mathbb{G}_m$$
where $\mathrm{ord}(\frac{a_3}{a_4})\geq\mathrm{ord}(-\frac{c_4}{c_3})=\mathrm{ord}(\frac{c_4}{c_3})$, then there exists $n\in\mathbb{Z}$ such that
$$\frac{a_3^n}{a_4^n}=-\frac{c_4}{c_3},\quad \text{i.e.,}\quad a_3^nc_3=-a_4^nc_4.$$
We are reduced to the previous case. In particular, the abelian group $\langle g_3,\varepsilon^n g_4\rangle$ has no fixed points on $\mathbb{P}^3$.

(4) When $c_1=-c_2$, $c_3\neq\pm c_4$ and G_T contains an element
$$\varepsilon=\mathrm{diag}(1,-1,a_3,a_4),\quad a_3,a_4\in\mathbb{G}_m(k),\quad a_3\neq\pm a_4$$
where $\mathrm{ord}(\frac{c_4}{c_3})>\mathrm{ord}(\frac{a_3}{a_4})=-\mathrm{ord}(\frac{a_3}{a_4})$, then there exists $n\in\mathbb{Z}$ such that
$$\frac{c_4^{2n}}{c_3^{2n}}=-\frac{a_3}{a_4},$$
and thus
$$\varepsilon\cdot g_4^{2n}=\mathrm{diag}(1,-1,a_3c_3^{2n},-a_3c_3^{2n}).$$
By the observation above, the abelian group $\langle g_3g_4,\varepsilon g_4^{2n}\rangle$ does not fix points on $\mathbb{P}^3$.

Thus, Condition **(A)** implies **(U)** when $\pi^*(G)=\mathfrak{K}_7$.

Case $\mathfrak{K}_8$: The group is given by

$$\mathfrak{K}_8=\left\langle\begin{pmatrix}1&1&1\\0&0&-1\\0&-1&0\end{pmatrix},\begin{pmatrix}0&0&1\\-1&-1&-1\\1&0&0\end{pmatrix}\right\rangle.$$

This is a subgroup contained in a $\mathfrak{D}_4$, covered in Lemma 5.9.

Case $\mathfrak{K}_9$: This is the exceptional case. The group is given by

$$\mathfrak{K}_9 = \left\langle \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \right\rangle.$$

In Section 6, we show that $\beta(X, G) \neq 0$ for all G with $\pi^*(G) = \mathfrak{K}_9$.

□

Lemma 5.9. *Assume that* $\pi^*(G) \subseteq \mathfrak{D}_4 = \langle \tau_1, \tau_2 \rangle$, *where*

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then

$$(\mathbf{A}) \Longleftrightarrow (\mathbf{U}).$$

Proof. We first assume that $\pi^*(G) = \langle \tau_1, \tau_2 \rangle$. As explained in Section 4, a simpler smooth projective model X in this case is the blowup of a quadric cone at its vertex. In particular, we have

$$\mathrm{Pic}(X) = \mathbb{Z} \oplus \mathrm{P}, \quad \mathrm{P} = \mathbb{Z} \oplus \mathbb{Z}$$

with G acting trivially on the first summand, and switching two factors of the second summand P.

Let Σ be the fan of X given in Section 4. We note that $\pi^*(G)$ acts trivially on the 1-dimensional sublattice $\mathrm{N}' \subset \mathrm{N}$ spanned by the ray $v_1 = (-1, 0, -1)$. Let $\sigma \in \Sigma$ be the cone generated by v_5. It corresponds to a G-invariant toric boundary divisor $D_\sigma \subset X$. On the other hand, the sublattice N' also gives rise to a quotient torus, cf. [14, Section 2.3]. In particular, we have

$$(\mathrm{N}/\mathrm{N}')^\vee = \sigma^\perp \cap \mathrm{M} \simeq \mathbb{Z}^2.$$

For any cone $\sigma' \in \Sigma$ such that $\sigma' \supseteq \sigma$, put

$$\bar{\sigma}' := (\sigma' + \mathbb{R}\sigma)/\mathbb{R}\sigma \subset (\mathrm{N}/\mathrm{N}')_{\mathbb{R}}.$$

All such $\bar{\sigma}'$ form a new G-invariant fan Σ_σ. Let $X(\Sigma_\sigma)$ be the toric variety associated with Σ_σ. One can check that $X(\Sigma_\sigma) = \mathbb{P}^1 \times \mathbb{P}^1$.

By [14, Section 2.3], $X(\Sigma_\sigma)$ is G-isomorphic to D_σ, and there exists a G-equivariant rational map

$$\rho : X \dashrightarrow X(\Sigma_\sigma) \simeq D_\sigma.$$

Note that the G-action on D_σ is not necessarily generically free. The map ρ induces a homomorphism of G-lattices

$$\rho^* : \mathrm{Pic}(D_\sigma) \to \mathrm{Pic}(X).$$

This yields a commutative diagram of G-modules

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{M} \arrow[r] & \mathrm{PL}(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r] & 0 \\ 0 \arrow[r] & (\mathrm{N}/\mathrm{N}')^\vee \arrow[r] \arrow[u] & \mathrm{PL}(D_\sigma) \arrow[r] \arrow[u] & \mathrm{Pic}(D_\sigma) \arrow[r] \arrow[u, "\rho^*"'] & 0 \end{tikzcd}$$

where $\mathrm{PL}(X) = \mathbb{Z}^6$ and $\mathrm{PL}(D_\sigma) = \mathbb{Z}^4$. Following the diagram, one sees that the dual map

$$(\rho^*)^\vee : \mathrm{Pic}(X)^\vee \to \mathrm{Pic}(D_\sigma)^\vee$$

can be identified with the canonical projection (note that $\mathrm{Pic}(X)$ is self-dual under the G-action)

$$\mathrm{P} \oplus \mathbb{Z} \to \mathrm{P}.$$

Now assume that the G-action on X satisfies Condition **(A)**. Let

$$\beta := \beta(X, G) \in \mathrm{H}^2(G, \mathrm{Pic}(X)^\vee \otimes k^\times),$$

and H be the maximal subgroup of G such that $\pi^*(H) = \langle \tau_1, \tau_2^2 \rangle \simeq C_2^2$. We have that $[G : H] = 2$ and H acts trivially on P; in particular,

$$\mathrm{P} = \mathrm{Ind}_H^G(\mathbb{Z})$$

is the G-module induced from the trivial H-module $\mathbb{Z}$. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{H}^2(G, (\mathrm{P} \oplus \mathbb{Z}) \otimes k^\times) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{res}_1"] & \mathrm{H}^2(G, \mathrm{P} \otimes k^\times) \arrow[d, "\mathrm{res}_2"] \\ \mathrm{H}^2(H, (\mathrm{P} \oplus \mathbb{Z}) \otimes k^\times) \arrow[r, "\mathrm{pr}_2"] & \mathrm{H}^2(H, \mathrm{P} \otimes k^\times) \end{tikzcd}$$

where res_1 and res_2 are the corresponding restriction homomorphisms, and pr_1 and pr_2 are projections induced by $(\rho^*)^\vee$. By functoriality,

$$\mathrm{pr}_1(\beta) = \beta(D_\sigma, G)$$

where $\beta(D_\sigma, G)$ is the class corresponding to the G-action on D_σ.

Since $\pi^*(H) = C_2^2$ is conjugate to $\mathfrak{K}_7$, by the proof of Lemma 5.8, Condition **(A)** implies that the H-action on X is **(U)**, and thus

$$\mathrm{res}_1(\beta) = 0, \quad \mathrm{pr}_2(\mathrm{res}_1(\beta)) = 0.$$

By Lemma 2.3, we know that res_2 is injective. It follows that

$$\mathrm{pr}_1(\beta) = 0$$

and thus the G-action on D_σ is **(U)**. Now let

$$\varrho : X \to \bar{X}$$

be the contraction of the boundary divisor in X corresponding to the ray $v_6 = (1, 0, 1)$. Then $\bar{X} \subset \mathbb{P}^4$ is a cone over a smooth quadric surface, and ϱ is the blowup of its vertex. The strict transform $\rho_*(D_\sigma)$ is a G-equivariantly unirational surface in $\bar{X}$.

Finally, the same argument as in [5, Proposition 3.1] shows that the G-action on $\bar{X}$ is **(U)**: we have a G-equivariant dominant rational map

$$\varrho_*(D_\sigma) \times \mathbb{P}^4 \dashrightarrow \bar{X},$$

sending the pair of points $(q_1, q_2) \in \varrho_*(D_\sigma) \times \mathbb{P}^4$ to the second intersection point of X with the line passing through q_1 and q_2. It follows that the G-action on X is also **(U)**.

The same proof applies when $\pi^*(G)$ is a subgroup of $\mathfrak{D}_4 = \langle \tau_1, \tau_2 \rangle$ and G swaps the two factors of P. When G does not swap the two factors, $\pi^*(G)$ has been already covered by previous lemmas. $\square$

Lemma 5.10. *Assume that* $\pi^*(G) = \mathfrak{D}_4$. *Then*

$$(\mathbf{A}) \Longleftrightarrow (\mathbf{U}).$$

Proof. There are 8 conjugacy classes of $\mathfrak{D}_4$ in $\mathsf{GL}_3(\mathbb{Z})$: up to conjugation, two of them contain θ_2; two of them contain θ_3; among the rest, one contains $\mathfrak{K}_1$ and one contains $\mathfrak{K}_6$. From the analysis above, we know that Condition **(A)** fails for these 6 classes.

One of the two remaining classes is covered by Lemma 5.9. In the other case, $\pi^*(G)$ is generated by

$$\iota_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \theta_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

This is realized on $X = (\mathbb{P}^1)^3$, where

$$\mathrm{Pic}(X) = \mathbb{Z} \oplus \mathrm{P}, \quad \mathrm{P} = \mathbb{Z} \oplus \mathbb{Z},$$

ι_2 acts trivially on $\mathrm{Pic}(X)$ and θ_1 switches the two factors of P. Since $\pi^*(G)$ contains a subgroup conjugated in $\mathsf{GL}_3(\mathbb{Z})$ to $\mathfrak{K}_4$, we know that

$$G_T \subset \{(t, 1, 1) : t \in \mathbb{G}_m(k)\} \subset T(k).$$

Let H be the subgroup of G generated by G_T and lifts to G of ι_2 and θ_1^2. It follows that H is abelian and $[G : H] = 2$. Thus,

$$\mathrm{B}^2(G, k^\times) = 0,$$

and Condition **(A)** implies that

$$\beta(X, G) \in \mathrm{B}^2(G, k^\times \otimes \mathrm{Pic}(X)) \simeq \mathrm{B}^2(G, k^\times \otimes \mathrm{P}).$$

Note that H acts trivially on $\mathrm{Pic}(X)$ and $\mathrm{P} = \mathrm{Ind}_H^G(\mathbb{Z})$ for the trivial G-module $\mathbb{Z}$. Since H is abelian, we have that $\mathrm{B}^2(H, k^\times \otimes \mathrm{P}) = 0$.

Lemma 2.3 shows that $\mathrm{B}^2(G, k^\times \otimes \mathrm{P}) = 0$. Thus, we conclude that $\beta(X, G) = 0$. ☐

6. The exceptional case $\mathfrak{K}_9$

This section is devoted to a proof of the following lemma, which completes the proof of Lemma 5.8.

Lemma 6.1. *Assume that $G_T \neq 1$ and $\pi^*(G)$ contains a subgroup conjugated in $\mathsf{GL}_3(\mathbb{Z})$ to*

$$\mathfrak{K}_9 = \left\langle \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \right\rangle.$$

Then the G-action on a smooth projective model X fails **(U)**.

Proof. We may assume that $\pi^*(G) = \mathfrak{K}_9 \simeq C_2^2$, and no proper subgroup of G surjects to C_2^2 via π^*. Then G is generated by

$$\sigma_1 : (t_1, t_2, t_3) \mapsto (b_1 t_2, b_2 t_1, \frac{b_3}{t_1 t_2 t_3}), \quad \sigma_2 : (t_1, t_2, t_3) \mapsto (\frac{c_1}{t_1 t_2 t_3}, c_2 t_3, c_3 t_2).$$

The torus part G_T is generated by

$$\sigma_1^2 = \mathrm{diag}(b_1 b_2, b_1 b_2, \frac{1}{b_1 b_2}), \quad \sigma_2^2 = \mathrm{diag}(\frac{1}{c_2 c_3}, c_2 c_3, c_2 c_3),$$

and

$$(\sigma_1 \sigma_2)^2 = \mathrm{diag}(\frac{c_1 c_3}{b_1 b_3}, \frac{b_1 b_3}{c_1 c_3}, \frac{c_1 c_3}{b_1 b_3}).$$

Since G is finite, we know that $b_1 b_2, c_2 c_3$ and $c_1 c_3 / b_1 b_3$ have finite orders. Up to a change of variables

$$t_1 \mapsto r_1 t_1, \quad t_2 \mapsto r_2 t_2, \quad t_3 \mapsto r_3 t_3,$$

where $r_1, r_2, r_3 \in k^\times$ are such that

$$r_1 b_1 = r_2, \quad r_1 r_2 r_3^2 b_3 = r_1^2 r_2 r_3 c_1 = 1,$$

we may assume that $b_1 = b_3 = c_1 = 1$, and b_2, c_2, c_3 are roots of unity whose orders are powers of 2.

When the G-action satisfies Condition **(A)**, we know that G_T is cyclic. Indeed, if G_T is not cyclic, then G contains one of the following subgroups which fail Condition **(A)**:

$$\langle \sigma_1, \mathrm{diag}(1, 1, -1) \rangle, \quad \langle \sigma_1, \mathrm{diag}(-1, -1, 1) \rangle,$$
$$\langle \sigma_2, \mathrm{diag}(-1, 1, 1) \rangle \quad \langle \sigma_2, \mathrm{diag}(1, -1, -1) \rangle,$$
$$\langle \sigma_1 \sigma_2, \mathrm{diag}(1, -1, 1) \rangle \quad \langle \sigma_1 \sigma_2, \mathrm{diag}(-1, 1, -1) \rangle.$$

From this, we see that at least two of

$$\sigma_1^2, \quad \sigma_2^2, \quad (\sigma_1 \sigma_2)^2$$

have order 1 or 2. Up to a permutation of coordinates, we may assume that the latter two have order 1 or 2. One can check that the G-action then satisfies Condition **(A)**. Let $n \in \mathbb{Z}$ such that b_2 has order 2^{n-2}. There are three cases:

(1) $c_2 = 1, c_3 = -1$: in this case $G \simeq \mathfrak{Q}_{2^n}$,
(2) $c_2 = c_3 = 1$: in this case $G \simeq \mathfrak{D}_{2^{n-1}}$,
(3) $c_2 = c_3 = -1$: in this case $G \simeq \mathfrak{SD}_{2^n}$.

In each case, we have $\sigma_1 = x$ and $\sigma_2 = y$, where x, y are the same as in the presentations of G, with generators and relations, given in Section 2. Using the resolutions (2.8), (2.9), and (2.10), we compute $\beta(X, G)$, as an element in $\mathrm{H}^3(G, \mathrm{Pic}(X)^\vee)$, following the recipe in Section 2.

We recall the presentation of $\mathrm{Pic} := \mathrm{Pic}(X)$ on the smooth projective model **(S)**, via the exact sequence

$$0 \to \mathrm{M} \to \mathrm{PL} \to \mathrm{Pic} \to 0,$$

where $\mathrm{PL} = \mathbb{Z}^{14}$ is generated by the following 14 rays:

$$(-1,0,0),(-1,1,0),(0,-1,1),(0,0,-1),(0,0,1),(0,1,-1),(1,-1,0),$$
$$(1,0,0),(1,0,-1),(1,-1,1),(0,-1,0),(0,1,0),(-1,1,-1),(-1,0,1),$$

labeled by $v_i, i = 1, \dots, 14$, in order. The character lattice M is embedded in PL as a submodule with basis

$$\begin{aligned} m_1 &= -v_1 - v_2 + v_7 + v_8 + v_9 + v_{10} - v_{13} - v_{14}, \\ m_2 &= v_2 - v_3 + v_6 - v_7 - v_{10} - v_{11} + v_{12} + v_{13}, \\ m_3 &= v_3 - v_4 + v_5 - v_6 - v_9 + v_{10} - v_{13} + v_{14}. \end{aligned}$$

Let $p_1 \dots, p_{11}$ be a basis of Pic, and $e_1, \dots, e_{11}$ the corresponding dual basis of $\mathrm{Pic}^\vee$, such that p_i can be lifted to PL by

$$p_i \mapsto v_{4+i}, \quad i = 1, 2, \dots, 9, \tag{6.1}$$
$$p_{10} \mapsto (v_4 - v_5), \quad p_{11} \mapsto (-v_4 + v_5 - v_9 + v_{14}).$$

Observe that the resolutions (2.8), (2.9), and (2.10) start with the same first step. Indeed, the images of $\mathrm{id}_{\mathrm{Pic}}$ in $\mathrm{H}^1(G, \mathrm{M} \otimes \mathrm{Pic}^\vee)$ are the same in all three cases of G. We first compute this intermediate class via

$$\begin{tikzcd} \mathrm{id}_{\mathrm{Pic}} \in \mathrm{Pic} \otimes \mathrm{Pic}^\vee & \mathrm{PL} \otimes \mathrm{Pic}^\vee \arrow[l] \arrow[d, "\begin{pmatrix} 1-x & 1-y \end{pmatrix}"] & & \\ & (\mathrm{PL} \otimes \mathrm{Pic}^\vee)^2 & (\mathrm{M} \otimes \mathrm{Pic}^\vee)^2 \arrow[l, hook'] & (k(T)^\times \otimes \mathrm{Pic}^\vee)^2 \arrow[l] \end{tikzcd} \tag{6.2}$$

We choose a lift of $\mathrm{id}_{\mathrm{Pic}}$ to $\mathrm{PL} \otimes \mathrm{Pic}^\vee$ given by (6.1). The resulting class in $(\mathrm{M} \otimes \mathrm{Pic}^\vee)^2$ is

$$((-m_2 - m_3) \otimes e_3, \quad (m_2 + m_3) \otimes e_1 + m_2 \otimes e_2 + m_1 \otimes e_4 + \\ + (-m_2 - m_3) \otimes e_{10} + (m_2 + m_3) \otimes e_{11}). \quad (6.3)$$

This class represents the image of $\mathrm{id}_{\mathrm{Pic}}$ in $\mathrm{H}^1(G, \mathrm{M} \otimes \mathrm{Pic}^\vee)$. We lift this class to $(\mathrm{M} \otimes \mathrm{Pic}^\vee)^2$ via the set-theoretic map $\mathrm{M} \to k(T)^\times$ given by

$$(a_1, a_2, a_3) \to t_1^{a_1} t_2^{a_2} t_3^{a_3}.$$

The next step of the computation depends on the isomorphism class of G. We proceed case-by-case.

For $G = \mathfrak{Q}_{2^n}$, we continue (6.2) via

$$\begin{tikzcd}
(k(T)^\times \otimes \mathrm{Pic}^\vee)^2 \arrow[r, "\begin{pmatrix} N_x & yx+1 \\ -1-y & x-1 \end{pmatrix}"] & (k(T)^\times \otimes \mathrm{Pic}^\vee)^2 & \\
& (k^\times \otimes \mathrm{Pic}^\vee)^2 \arrow[u, hook] & \\
& (\mathbb{Q} \otimes \mathrm{Pic}^\vee)^2 \arrow[u] \arrow[r, "\begin{pmatrix} 1-x \\ yx-1 \end{pmatrix}"] & \mathbb{Q} \otimes \mathrm{Pic}^\vee \\
& & \mathbb{Z} \otimes \mathrm{Pic}^\vee \arrow[u, hook]
\end{tikzcd}$$

The first two arrows come from the resolution (2.8). The rest of the diagram arises from the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

Since G is a finite group, we may replace $k^\times$ by $\mathbb{Q}/\mathbb{Z}$, via the map

$$\{x \in k^\times : \text{the order of } x \text{ is finite}\} \to \mathbb{Q}/\mathbb{Z}, \quad x \mapsto \frac{\log(x)}{2\pi i}.$$

The class $\beta(X, G) \in \mathrm{H}^2(G, k^\times \otimes \mathrm{Pic}^\vee)$ is represented by the image of (6.3) in $(k^\times \otimes \mathrm{Pic}^\vee)^2$ via the diagram above. This image is

$$(-1 \otimes (e_1 + e_2 + e_4 + e_{10} + e_{11}) + (b_2^{-2^{n-3}}) \otimes e_3, \\ b_2 \otimes (e_4 + e_{10} - e_3 - e_{11}) + (-b_2^{-1}) \otimes e_1). \quad (6.4)$$

To determine whether or not this class vanishes, we map it further to $\mathrm{H}^3(G, \mathbb{Z} \otimes \mathrm{Pic}^\vee)$ using the sequence above. In particular, we choose a lift $k^\times \to \mathbb{Q}$ such that

$$-1 \mapsto \frac{1}{2}, \quad b_2 \mapsto \frac{\log(b_2)}{2\pi i}, \quad -b_2^{-1} \mapsto \frac{1}{2} - \frac{\log(b_2)}{2\pi i},$$

$$b_2^{-2^{n-3}} \mapsto -\frac{\log(b_2)}{2\pi i}\cdot 2^{n-3}, \qquad \text{where} \qquad 0 \le \frac{\log(b_2)}{2\pi i} < 1.$$

Using this choice, the image of (6.4) in $\mathbb{Z}\otimes \mathrm{Pic}^\vee = \mathrm{Pic}^\vee$, following the diagram above, is given by

$$\beta = (-1,0,1,0,0,0,-1,0,0,1,0),$$

under the basis $e_1,\dots,e_{11}$. On the other hand, the image of

$$\nu : (\mathrm{Pic}^\vee)^2 \xrightarrow{\begin{pmatrix} 1-x \\ yx-1 \end{pmatrix}} \mathrm{Pic}^\vee$$

is the $\mathbb{Z}[G]$-module generated by

$$(0,0,0,0,0,0,2,0,0,-2,0),$$
$$(1,0,0,0,0,0,1,0,0,-2,1), \quad (0,1,0,0,0,0,1,-1,0,-1,1),$$
$$(0,0,1,0,0,0,1,0,0,-2,1), \quad (0,0,0,1,0,0,1,-1,0,-1,1)$$
$$(0,0,0,0,1,0,1,-1,0,-2,0), \quad (0,0,0,0,0,1,0,0,-1,0,0).$$

One can check that $\beta \notin \mathrm{im}(\nu)$. It follows that $\beta(X,G) \neq 0$.

Similarly, when $G = \mathfrak{D}_{2^{n-1}}$, using its resolution (2.9), we continue (6.2) via

$$\begin{tikzcd}
(k(T)^\times \otimes \mathrm{Pic}^\vee)^2 \arrow[r, "{\begin{pmatrix} N_x & 1+yx & 0 \\ 0 & x-1 & 1+y \end{pmatrix}}"] & (k(T)^\times \otimes \mathrm{Pic}^\vee)^3 & \\
& (k^\times \otimes \mathrm{Pic}^\vee)^3 \arrow[u, hook] & \\
& (\mathbb{Q}\otimes \mathrm{Pic}^\vee)^3 \arrow[u] \arrow[r, "{\begin{pmatrix} 1-x & y+1 & 0 & 0 \\ 0 & -N_x & 1-yx & 0 \\ 0 & 0 & 1-x & 1-y \end{pmatrix}}"] & (\mathbb{Q}\otimes \mathrm{Pic}^\vee)^4 \\
& & (\mathrm{Pic}^\vee)^4 \arrow[u, hook]
\end{tikzcd}$$

The class $\beta(X,G)$ is represented in $(k^\times \otimes \mathrm{Pic}^\vee)^3$ by

$$\left(b_2^{-2^{n-2}} \otimes e_3, \qquad b_2 \otimes (e_4 + e_{10} - e_1 - e_3 - e_{11}), \quad e\right),$$

where e is the identity element of $k^\times \otimes \mathrm{Pic}^\vee$. We choose a lift $k^\times \to \mathbb{Q}$ such that

$$b_2 \mapsto \frac{\log(b_2)}{2\pi i}, \quad b_2^{-2^{n-2}} \mapsto -\frac{\log(b_2)}{2\pi i}\cdot 2^{n-2},$$

where

$$0 \le \frac{\log(b_2)}{2\pi i} < 1.$$

With this choice, the image of $\beta(X,G)$ in $\mathrm{H}^3(G,\mathrm{Pic}^\vee)$ is represented in $(\mathrm{Pic}^\vee)^4$ by

$$\beta = (\mathbf{0}, (0, -\frac{\log(b_2)}{2\pi i}\cdot 2^{n-2}, 0, -\frac{\log(b_2)}{2\pi i}\cdot 2^{n-2}, 0,0,0,0,0,0,0), \mathbf{0}, \mathbf{0}, \mathbf{0}),$$

where $\mathbf{0}$ denotes the zero element in $\mathrm{Pic}^\vee$. Note that

$$-\frac{\log(b_2)}{2\pi i}\cdot 2^{n-2}$$

is an odd integer since $\mathrm{ord}(b_2) = 2^{n-2}$. On the other hand, the intersection of the image of

$$\nu : (\mathrm{Pic}^\vee)^3 \xrightarrow{\begin{pmatrix} 1-x & y+1 & 0 & 0 \\ 0 & -N_x & 1-yx & 0 \\ 0 & 0 & 1-x & 1-y \end{pmatrix}} (\mathrm{Pic}^\vee)^4$$

with the subspace $\langle \mathbf{0}\rangle \times \langle \mathbf{0}\rangle \times \mathrm{Pic}^\vee \times \langle \mathbf{0}\rangle$ is generated by the following elements in $\mathrm{Pic}^\vee$:

$$(1,0,1,0,0,0,0,0,0,0,0), \quad (0,1,0,1,1,0,1,1,0,0,0),$$
$$(0,0,0,0,2,0,2,2,0,0,0), \quad (0,0,0,0,0,2,0,0,2,0,0).$$

One can check that $\beta \notin \mathrm{im}(\nu)$ and thus $\beta(X,G)\neq 0$.

Finally, for $G = \mathfrak{SD}_{2^n}$, using its resolution (2.10), we continue (6.2) via

$$\begin{tikzcd}
(k(T)^\times \otimes \mathrm{Pic}^\vee)^2 \arrow[r, "\begin{pmatrix} L_2 & 0 \\ L_1 & y+1 \end{pmatrix}"] & (k(T)^\times \otimes \mathrm{Pic}^\vee)^2 & \\
& (k^\times \otimes \mathrm{Pic}^\vee)^2 \arrow[u, hook] & \\
& (\mathbb{Q}\otimes \mathrm{Pic}^\vee)^2 \arrow[u] \arrow[r, "\begin{pmatrix} -L_3 & 0 \\ L_4 & 1-y \end{pmatrix}"] & (\mathbb{Q}\otimes \mathrm{Pic}^\vee)^2 \\
& & (\mathrm{Pic}^\vee)^2 \arrow[u, hook]
\end{tikzcd}$$

The class $\beta(X,G)$ is represented in $(k^\times \otimes \mathrm{Pic}^\vee)^2$ by

$$(b_2^{2^{n-4}-1}\otimes e_1 + b_2^{2^{n-4}}\otimes e_2 + b_2^{2^{n-4}+1}\otimes (e_4 - e_3) + b_2(e_{10}-e_{11}), \quad -1\otimes e_2).$$

We choose a lift of $k^\times \to \mathbb{Q}$ such that

$$-1 \mapsto \frac{1}{2}, \quad b_2^r \mapsto \frac{\log(b_2)}{2\pi i}\cdot r, \quad \forall r\in\mathbb{Z},$$

where

$$0 \leq \log(b_2) < 2\pi i.$$

Under this lift, the image of $\beta(X, G)$ in $\mathrm{H}^3(G, \mathrm{Pic}^\vee)$ is represented in $(\mathrm{Pic}^\vee)^2$ by

$$\beta = ((0, 1, 0, -1, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)).$$

We find that β is not in the image of

$$\nu : (\mathrm{Pic}^\vee)^2 \xrightarrow{\begin{pmatrix} -L_3 & 0 \\ L_4 & 1-y \end{pmatrix}} (\mathrm{Pic}^\vee)^2.$$

Indeed, one can check that β is not in the intersection $\mathrm{im}(\nu) \cap (\mathrm{Pic}^\vee \times \mathbf{0})$, which is the $\mathbb{Z}[G]$-module generated by

$$((1, 1, -1, -1, 1, 0, 1, -1, 0, -2, 0), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)),$$

$$((0, 2, 0, -2, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)).$$

We conclude that $\beta(X, G) \neq 0$. This completes the proof of Lemma 6.1. □

Example 6.2. Let $G = \mathfrak{D}_4$ be generated by

$$(t_1, t_2, t_3) \mapsto (t_2, -t_1, \frac{1}{t_1 t_2 t_3}), \quad (t_1, t_2, t_3) \mapsto (\frac{1}{t_1 t_2 t_3}, t_3, t_2).$$

Here, $G_T = \langle(-1, -1, -1)\rangle \simeq C_2$ and $\pi^*(G) = \mathfrak{K}_9$. The G-action satisfies **(A)** – the two noncyclic $\mathfrak{K}_4 \subset \mathfrak{D}_4$ map to C_2 via π, and fix points on the smooth model $(\mathbb{P}^1)^3$. However, $\beta(X, G)$ does not vanish, and the G-action fails **(U)**.

Remark 6.3. Using the analysis in Section 5, one can check that when $\pi^*(G)$ strictly contains $\mathfrak{K}_9$, the G-action fails Condition **(A)**.

7. Stable linearizability

In this section, we prove Theorem 1.1, i.e., a criterion for unirationality and stable linearizability of generically free G-actions on toric threefolds. We assume the necessary Condition **(A)**.

Step 1. By Theorem 2.4, for smooth projective toric varieties X, unirationality of the G-action is equivalent to the vanishing of the class

$$\beta(X, G) \in \mathrm{H}^2(G, \mathrm{Pic}(X)^\vee \otimes k^\times) = \mathrm{H}^3(G, \mathrm{Pic}(X)^\vee).$$

By Lemma 2.1, this class vanishes if and only if it vanishes upon restriction to every p-Sylow subgroup of G.

Step 2. When $p \neq 2, 3$, the p-Sylow subgroup of $G \subset \mathrm{Aut}(T)$ is a subgroup of translations $T(k) \subset \mathrm{Aut}(T)$, see (2.1). Since it has fixed points in the boundary $X \setminus T$, the action is unirational, by (2.6).

Step 3. For $p = 3$, Proposition 5.1 implies that unirationality is equivalent to Condition **(A)**.

Step 4. For $p = 2$, Proposition 5.2 characterizes unirationality, as stated in Theorem 1.1. Note that a G-action with $\pi^*(G)$ as in (1.1) fails Condition **(SP)**.

Step 5. Stable linearizability of the action is governed by Theorem 2.5: assuming unirationality, **(SL)** follows from the stable permutation property of $\mathrm{Pic}(X)$, as a G-module. This property only depends on the image $\pi^*(G)$ in $\mathsf{GL}(\mathrm{N})$. The corresponding actions have been analyzed in [17]: the G-action on $\mathrm{Pic}(X)$ fails **(SP)** if and only if $\pi^*(G)$ contains one of the three 2-groups indicated in Theorem 1.1.

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