EQUIVARIANT GEOMETRY OF CUBIC THREEFOLDS WITH NON-ISOLATED SINGULARITIES

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Abstract. We study linearizability of actions of finite groups on cubic threefolds with non-isolated singularities.

1. Introduction

Let k be an algebraically closed field of characteristic zero, X a smooth projective variety over k of dimension n and $G \subseteq \mathrm{Aut}(X)$ a subgroup of automorphisms. The G-action on X is *linearizable* if it is equivariantly birational to a linear G-action on $\mathbb{P}^n$, and *unirational* if X is G-equivariantly dominated by the projectivization of a linear representation of G.

In [7], [6], [5] and [8], we have addressed the problems of unirationality and linearizability of actions of finite groups on cubic threefolds with *isolated* singularities. In this paper, we extend the study of birational properties of generically free regular actions of finite groups on singular cubic threefolds to those with *non-isolated* singularities.

Detailed knowledge of degenerations of cubic threefolds, together with their automorphisms, plays an important role in moduli theory, see, e.g., [1], [4]. Indeed, the classification of isolated singularities in [18] was one of the motivations of our work. Another source of inspiration comes from arithmetic and birational geometry over nonclosed fields, as in [10] or [9].

We proceed to describe our strategy: we start by identifying all possibilities for actions and the corresponding normal forms. Roughly, cubic threefolds with non-isolated singularities are of four types, according to the geometry of the singular locus: line, conic, plane, twisted quartic. More precisely, denote by $\mathrm{Sing}(X)$ the singular locus of X. Suppose that $\dim(\mathrm{Sing}(X)) \geq 1$, and X is not a cone. We observe that the secant variety of $\mathrm{Sing}(X)$ is either X or has dimension ≤ 2; otherwise,

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X is reducible. Building on this, the possibilities of $\mathrm{Sing}(X)$ have been classified in [19, Proposition 4.2]:

- When $\dim(\mathrm{Sing}(X)) \geq 2$, then $\mathrm{Sing}(X)$ is a plane.
- When $\dim(\mathrm{Sing}(X)) = 1$ and $\mathrm{Sing}(X)$ contains a plane curve, then the union of one-dimensional components of $\mathrm{Sing}(X)$ is either a line, a smooth conic, or two lines intersecting at a point.
- When $\dim(\mathrm{Sing}(X)) = 1$ and $\mathrm{Sing}(X)$ contains a curve not contained in a plane, then $\mathrm{Sing}(X)$ is a smooth rational normal quartic curve and there is a unique such X, known as the *chordal cubic*, given by

$$x_1x_4^2 + x_2^2x_5 - x_1x_3x_5 - 2x_2x_3x_4 + x_3^3 = 0.$$

We focus on actions not fixing singular points on X, since otherwise, the actions are linearizable. The resulting cases are analyzed using a variety of tools from birational geometry, including the connection between equivariant geometry and geometry over nonclosed fields in [12]. We also restrict to finite group actions. Our principal results are:

- All actions on cubics singular along a line are linearizable, by Theorem 3.1.
- All actions on the chordal cubic are linearizable, by Theorem 5.1.
- All actions on cubics singular along a plane are linearizable, by Theorem 6.2.
- Most actions on cubics singular along a conic are not linearizable, by Theorem 4.3. The linearizability problem of some actions in this case remains open; see Section 4 for more details.
- All actions on cubic threefolds with non-isolated singularities are unirational, by Theorem 4.5.

One of the corollaries of classifications in [1], [18], [19], [16] is the following: if $X \subset \mathbb{P}^4$ is a K-unstable rational cubic threefold with isolated singularities and $G \subseteq \mathrm{Aut}(X)$ is a finite subgroup, then the G-action on X is linearizable. This is no longer true for cubic threefolds with non-isolated singularities – there are non-linearizable G-actions on K-unstable cubic 3-folds singular along a conic; this is also the most interesting case in arithmetic considerations in [9].

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2. Tools from birational geometry

We recall the basic terminology: a G-action on $\mathbb{P}^n$ is called *linear* if it arises from projectivization $\mathbb{P}(V)$ of a G-representation V. We *do not* assume that the action on $\mathbb{P}(V)$ is generically free. We will use the following general results.

Proposition 2.1. *Let $X = \mathbb{P}_B(\mathcal{E})$ be the projectivization of a vector bundle $\mathcal{E}$ of rank $n+1$ over a smooth projective irreducible variety B, and $\pi : X \to B$ the associated $\mathbb{P}^n$-bundle. Assume that X carries a regular action of a finite group G such that $\mathcal{E}$ is G-linearized, and that the induced action on B is unirational. Then the action on X is G-unirational.*

Moreover, if the G-action on B is linearizable and generically free then X is linearizable.

Proof. By assumption, there exists a G-representation V such that G acts generically freely on $\mathbb{P}(V)$, which in turn admits a dominant G-equivariant rational map to B. Equivariantly resolving indeterminacy of this map $\widetilde{\mathbb{P}(V)} \to \mathbb{P}(V)$, we obtain the following G-equivariant diagram:

$$\begin{array}{ccc} \mathcal{E} & \longleftarrow & \tilde{\mathcal{E}} \\ \downarrow & & \downarrow \\ B & \longleftarrow & \widetilde{\mathbb{P}(V)} \end{array}$$

where $\tilde{\mathcal{E}}$ is the pullback of $\mathcal{E}$ to $\widetilde{\mathbb{P}(V)}$. By assumption, the G-action on $\mathbb{P}_B(\mathcal{E})$ lifts to $\mathcal{E}$. It follows that $\tilde{\mathcal{E}}$ is a G-linearized vector bundle over $\widetilde{\mathbb{P}(V)}$. By the no-name lemma, see, e.g., [13, Theorem 1], $\tilde{\mathcal{E}}$ is G-equivariantly birational to $\widetilde{\mathbb{P}(V)} \times \mathbb{A}^{n+1}$, with trivial action on the second factor, as the G-action on $\widetilde{\mathbb{P}(V)}$ is generically free. It follows that $X = \mathbb{P}_B(\mathcal{E})$ is dominated by $\mathbb{P}(W)$, for some G-representation W, thus is G-unirational.

The second claim is a projective version of the no-name lemma, see, e.g., [13, Theorem 1']. $\square$

Remark 2.2. This proposition applies in particular to actions on products of projective spaces, e.g., $\mathbb{P}^1 \times \mathbb{P}^2$ in Section 3, or to a nontrivial $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, in Section 5. In general, e.g, when the G-action on the base B is not generically free, or does not lift to $\mathcal{E}$, the linearization problem remains a challenge. The linearizability of actions on $(\mathbb{P}^1)^2$ was recently settled in [17]; the case of $(\mathbb{P}^1)^3$ is open.

The following proposition is a version of [8, Propositions 3.1 and 3.5].

Proposition 2.3. *Let k be an algebraically closed field of characteristic 0. Let $X \subset \mathbb{P}^n$, $n \geq 3$, be an irreducible cubic hypersurface over k which is not a cone, and $G \subseteq \mathrm{Aut}(X)$ a finite subgroup, acting linearly on $\mathbb{P}^n$. Assume that X contains a G-invariant subvariety S, which is G-unirational. Then X is G-unirational.*

Proof. By [12, Theorem 1.1], G-unirationality of X is equivalent to the following property: for every field K/k and every G-torsor T over K, the twist ${}^TX_K$ of X over K via T is K-unirational. Our assumption implies that all such twists are cubic hypersurfaces in $\mathbb{P}^n_K$, see [12, Lemma 10.1]. Every such twist contains a twisted form of S, defined over K. By assumption and [12, Theorem 1.1], this twisted form of S is unirational over K and, in particular, has K-rational points. Then ${}^TX_K$ also has K-rational points and is K-unirational by [15]. ☐

Remark 2.4. This proposition applies in particular when X contains a G-invariant linear subspace, e.g., a point, a line, or a plane; we use it in the proof of Theorem 4.5.

3. Line

Let $X \subset \mathbb{P}^4$ be an irreducible cubic threefold, singular along a line $\mathfrak{l}$. If $\mathrm{Sing}(X)$ contains another positive-dimensional component, then the action of $\mathrm{Aut}(X)$ on X is linearizable; indeed, the other component must be a line intersecting $\mathfrak{l}$ in a distinguished singular point. Hence we assume that $\mathfrak{l}$ is the only positive-dimensional component of $\mathrm{Sing}(X)$. The main theorem of this section is the following.

Theorem 3.1. *Let X be a cubic threefold, singular along a line, and $G \subseteq \mathrm{Aut}(X)$ a finite subgroup. Then the G-action on X is linearizable.*

A normal form for X is given by

$$x_1q_1(x_3,x_4,x_5) + x_2q_2(x_3,x_4,x_5) + c(x_3,x_4,x_5) = 0, \tag{3.1}$$

where $\mathfrak{l} = \{x_3 = x_4 = x_5 = 0\}$, q_1 and q_2 are quadratic forms, and c is a cubic form, see [19, Proposition 4.2]. We have a natural identification $\mathfrak{l} = \mathbb{P}^1_{x_1,x_2}$. Let $\beta : \tilde{\mathbb{P}} \to \mathbb{P}^4$ be the blowup of $\mathfrak{l}$. This yields the commutative diagram:

$$\begin{array}{ccc} & \tilde{\mathbb{P}} & \\ {}^{\beta}\swarrow & & \searrow^{\varpi} \\ \mathbb{P}^4 & \dashrightarrow & \mathbb{P}^2_{x_3,x_4,x_5} \end{array}$$

where ϖ is a $\mathbb{P}^2$-bundle, and the dashed arrow is the projection from $\mathfrak{l}$.

Let E be the β-exceptional divisor. Then β and ϖ induce a natural isomorphism

$$E \simeq \mathfrak{l} \times \mathbb{P}^2_{x_3,x_4,x_5} = \mathbb{P}^1_{x_1,x_2} \times \mathbb{P}^2_{x_3,x_4,x_5}.$$

Let $\tilde{X}$ be the strict transform of X on the fourfold $\tilde{\mathbb{P}}$. We have the induced $\mathrm{Aut}(X)$-equivariant commutative diagram:

$$\begin{array}{ccc} & \tilde{X} & \\ {}^{\beta|_{\tilde{X}}}\swarrow & & \searrow^{\pi} \\ X & \dashrightarrow & \mathbb{P}^2_{x_3,x_4,x_5} \end{array}$$

where $\pi = \varpi|_{\tilde{X}}$ is a morphism such that its fibers are isomorphic either to $\mathbb{P}^1$ or $\mathbb{P}^2$. Put $S := \tilde{X}|_E$. Then S is a divisor of bi-degree $(1,2)$ in $E \simeq \mathbb{P}^1_{x_1,x_2} \times \mathbb{P}^2_{x_3,x_4,x_5}$ that is given by

$$x_1 q_1(x_3,x_4,x_5) + x_2 q_2(x_3,x_4,x_5) = 0.$$

For general q_1 and q_2, S is a smooth del Pezzo surface of degree 5, the natural projection $S \to \mathbb{P}^2_{x_3,x_4,x_5}$ is a blowup of 4 points in general position, and the natural projection $S \to \mathbb{P}^1_{x_1,x_2}$ is a conic bundle with 3 singular fibers. If q_1 and q_2 are special, S may be singular.

First, assume that S is smooth. In this case, X can be given by

$$\begin{aligned} (3.2)\quad & x_1(x_3^2+\zeta_3 x_4^2+\zeta_3^2 x_5^2) + x_2(x_3^2+\zeta_3^2 x_4^2+\zeta_3 x_5^2)+ \\ & \quad + a_2(x_3^2x_4 + x_3x_4^2 + x_3^2x_5 + x_4^2x_5 - x_3^3 - x_4^3)+ \\ & \quad\quad + a_1x_3x_4x_5 + a_2x_3x_5^2 + a_3x_4x_5^2 + (a_4-2a_2)x_5^3 = 0, \end{aligned}$$

for some a_1, a_2, a_3, a_4, the morphism $\varpi|_S : S \to \mathbb{P}^2_{x_3,x_4,x_5}$ is a blowup of the points

$$(3.3)\qquad [1:1:1],\quad [1:1:-1],\quad [1:-1:1],\quad [-1:1:1],$$

and the singular fibers of the conic bundle $\beta|_S\colon S \to \mathfrak{l}$ lie over the points

$$[1:-1:0:0:0],\quad [1:-\zeta_3:0:0:0],\quad [1:-\zeta_3^2:0:0:0]. \tag{3.4}$$

Let $G \subseteq \mathrm{Aut}(X)$ be a finite subgroup. The set of points in (3.4) must be a G-orbit of length 3, unless one of them is fixed by G. Note that the image of G in $\mathrm{Aut}(\mathfrak{l}) \simeq \mathsf{PGL}_2(k)$ is contained in the subgroup isomorphic to $\mathfrak{S}_3$, generated by

$$(x_1, x_2) \mapsto (x_2, x_1) \tag{3.5}$$

and

$$(x_1, x_2) \mapsto (\zeta_3 x_1, \zeta_3^2 x_2).$$

Since this subgroup has an orbit of length 2 consisting of the points $[1:0]$ and $[0:1]$, we see that the points

$$[1:0:0:0:0],\quad [0:1:0:0:0]$$

form a G-orbit in $\mathfrak{l}$ of length 2, unless both are fixed by G. This allows us to classify all automorphism groups:

Proposition 3.2. *Let X be a cubic threefold singular along a line and such that the associated degree-5 del Pezzo surface is smooth. Assume that $\mathrm{Aut}(X)$ does not fix a singular point of X. Then, up to isomorphism, X is given by (3.2) and one of the following holds:*

(1) $a_1 \in k$, $a_2 = a_3 = a_4 = 1$, $\mathrm{Aut}(X) = \mathfrak{S}_3$ *is generated by*

$$\sigma_1 : (\mathbf{x}) \mapsto (x_2, x_1, x_3, x_5, x_4),$$
$$\sigma_2 : (\mathbf{x}) \mapsto (\zeta_3 x_1, \zeta_3^2 x_2, x_4, x_5, x_3).$$

(2) $a_1 = 1$, $a_2 = a_3 = a_4 = 0$, $\mathrm{Aut}(X) = \mathfrak{S}_4$ *is generated by* σ_1, σ_2 *and*

$$\iota_1 = \mathrm{diag}(1, 1, 1, -1, -1).$$

(3) $a_1 = a_2 = a_3 = a_4 = 0$, $\mathrm{Aut}(X) = \mathbb{G}_m \times \mathfrak{S}_4$ *is generated by* $\sigma_1, \sigma_2, \iota_1$ *and*

$$\tau_a = \mathrm{diag}(1, 1, a, a, a), \quad a \in k^\times.$$

Proof. By assumption, the induced G-action on $\mathfrak{l}$ is the $\mathfrak{S}_3$-action generated by (3.5). This action comes from $\mathrm{Aut}(X)$ only when

$$a_2 = a_3 = a_4.$$

There is an exact sequence

$$0 \to H \to G \to \mathfrak{S}_3 \to 0,$$

where H is the generic stabilizer of $\mathfrak{l}$, consisting of elements of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ b_1 & c_1 & d_1 & e_1 & f_1 \\ b_2 & c_2 & d_2 & e_2 & f_2 \\ b_3 & c_3 & d_3 & e_3 & f_3 \end{pmatrix}.$$

The condition that such elements leave invariant (3.2) gives a system of equations. Solving for the parameters b_i, c_i, d_i, e_i, f_i leads to the three cases in the assertion. □

We turn to the case when S is singular.

Proposition 3.3. *Let X be a cubic threefold singular along a line and such that the associated degree-5 del Pezzo surface S is singular. Assume that $\mathrm{Aut}(X)$ does not fix a singular point on X. Then, up to isomorphism, one of the following holds:*

(4) $X = \{x_1x_3^2+x_2x_4^2+x_5^3 = 0\}$, *and* $\mathrm{Aut}(X) = \mathbb{G}_m^2 \rtimes C_2$, *generated by*

$$\mathrm{diag}(t_1^{-2}, 1, t_1, 1, 1), \quad \mathrm{diag}(1, t_2^{-2}, 1, t_2, 1),$$

$$(\mathbf{x}) \mapsto (x_2, x_1, x_4, x_3, x_5), \quad t_1, t_2 \in k^\times.$$

(5) $X := \{x_1x_3^2+x_2x_4^2+x_3x_4x_5+x_5^3 = 0\}$, *and* $\mathrm{Aut}(X) = \mathbb{G}_m \rtimes C_2$, *generated by*

$$\mathrm{diag}(t^{-2}, t^2, t, t^{-1}, 1), \quad (\mathbf{x}) \mapsto (x_2, x_1, x_4, x_3, x_5), \quad t \in k^\times.$$

Proof. We start by observing that q_1 and q_2 from (3.1) are linearly independent, since S is not a cone. Now, we suppose that G does not fix points in $\mathfrak{l}$. In particular, G does not fix points in S.

If S has Du Val singularities, all possibilities for S are described in [11, Proposition 8.5]. In particular, since $\mathrm{Aut}(X)$ does not fix points in S, we see that S does not have a distinguished singular point, which leaves only one possibility for the singular locus of S – it consists of two singular point of type A_1. But in this case, the conic bundle $S \to \mathfrak{l}$ has exactly two singular fibers, one of them is not reduced, and the other is reduced and contained in the smooth locus of S, so they cannot be swapped by the action of G, which implies that G fixes two points in $\mathfrak{l}$.

Hence, the singularities of S are not Du Val. Thus, up to a change of coordinates, we may assume that q_1 and q_2 do not depend on x_5. Then, up to a change of variables, we may assume that

$$q_1 = x_3^2, \quad q_2 = x_4^2.$$

Now, the equation (3.1) takes the form

$$x_1x_3^2 + x_2x_4^2 + c(x_3, x_4, x_5) = 0.$$

After a linear change of x_1, x_2, we may assume that c does not contain monomials divisible by x_3^2 and x_4^2, so that

$$c(x_3, x_4, x_5) = c_1x_5^3 + x_5^2(c_2x_3 + c_3x_4) + c_4x_3x_4x_5.$$

- If $c_1 \neq 0$, we can change x_1, x_2 and x_5 to get

  $$c(x_3, x_4, x_5) = x_5^3 + c_4x_3x_4x_5.$$

  Now, if $c_4 = 0$, we get case (4). Otherwise we can scale coordinates so that $c_4 = 1$, and obtain case (5).
- If $c_1 = 0$ and $c_2 \neq 0$ or $c_3 \neq 0$, then we can change x_1, x_2 and x_5 to get

  $$c(x_3, x_4, x_5) = x_5^2(c_2x_3 + c_3x_4),$$

  where at least one of c_2 or c_3 is not zero, since X is not a cone. Then X has a distinguished singularity in $\mathfrak{l}$, so this point must be fixed by $\mathrm{Aut}(X)$. Hence, this case is impossible.
- If $c_1 = 0$ and c_2 and $c_3 = 0$, then $c(x_3, x_4, x_5) = c_4x_3x_4x_5$, which means that X is singular along a plane. Hence, this case is impossible.

Now we determine the automorphism groups of the two possible cases. In both cases, the group $\mathrm{Aut}(X)$ contains the infinite dihedral group $\mathbb{G}_m \rtimes C_2$, and $\mathbb{G}_m$ acts on $\mathfrak{l}$ with the fixed points

$$[1:0:0:0:0], \quad [0:1:0:0:0],$$

which are swapped by the action of C_2. Observe that X has $\mathsf{A}_2 \times \mathbb{A}^1$ singularity at every point of $\mathfrak{l}$ different from $[1:0:0:0:0]$ and $[0:1:0:0:0]$, but X has worse singularities at these two points, which implies that they form one $\mathrm{Aut}(X)$-orbit. It follows that there exists an exact sequence

$$0 \to H \to \mathrm{Aut}(X) \to \mathbb{G}_m \rtimes C_2 \to 0,$$

where H is the generic stabilizer of $\mathfrak{l}$. Similarly as in Proposition 3.2, a direct computation of H completes the proof. □

Consider the rational map $\chi\colon X \dashrightarrow \mathbb{P}^1_{x_1,x_2} \times \mathbb{P}^2_{x_3,x_4,x_5}$ given by

$$\mathbf{x} \mapsto \big((x_1, x_2), (x_3, x_4, x_5)\big),$$

for X in Propositions 3.2 and 3.3. The description of automorphisms implies that χ is $\mathrm{Aut}(X)$-equivariant. Following the labelling in Propositions 3.2 and 3.3, χ is a birational map in all five cases except case (3). We discuss linearizations:

(1) $\mathrm{Aut}(X) = \mathfrak{S}_3$, acting on $\mathbb{P}^1 \times \mathbb{P}^2$ diagonally via the usual action on $\mathbb{P}^1$ and the permutation action on $\mathbb{P}^2$. The action is linearizable, by Proposition 2.1.

(2) $\mathrm{Aut}(X) = \mathfrak{S}_4$, acting on $\mathbb{P}^1 \times \mathbb{P}^2$ via the $\mathfrak{S}_3$-action on $\mathbb{P}^1$ and the standard faithful linear action on $\mathbb{P}^2$; this is linearizable, by Proposition 2.1.

(3) In this case, χ is not birational. Instead, the $\mathrm{Aut}(X)$-equivariant birational map $\rho : X \dashrightarrow \mathbb{P}^3$ given by

$$(\mathbf{x}) \mapsto (x_2x_3, x_2x_4, x_2x_5, x_3^2 + \zeta_3 x_4^2 + \zeta_3^2 x_5^2)$$

yields linearizability. In detail, ρ factors through the birational maps π and φ:

$$X \overset{\varphi}{\dashrightarrow} X_{2,2} \overset{\pi}{\dashrightarrow} \mathbb{P}^3,$$

where $\varphi : X \dashrightarrow X_{2,2} \subset \mathbb{P}^5$ is the unprojection from the plane $\{x_4 = x_5 = 0\}$, given by

$$(\mathbf{x}) \mapsto (x_1x_2, x_2^2, x_3x_2, x_4x_2, x_5x_2, x_3^2 + \zeta_3 x_4^2 + \zeta_3^2 x_5^2),$$

and $X_{2,2} \subset \mathbb{P}^5_{y_1,\ldots,y_6}$ is the intersection of two quadrics given by

$$y_3^2 + \zeta_3 y_4^2 + \zeta_3^2 y_5^2 + y_2y_6 = y_3^2 + \zeta_3^2 y_4^2 + \zeta_3 y_5^2 + y_1y_6 = 0.$$

The singular locus of $X_{2,2}$ is the image of $\mathfrak{l}$, also a line. The birational map $\pi : X_{2,2} \dashrightarrow \mathbb{P}^3$ is the projection from this distinguished line, which fits the following commutative diagram:

$$\begin{array}{ccc} & \hat{X}_{2,2} & \\ \swarrow & & \searrow \\ X_{2,2} & \overset{\pi}{\dashrightarrow} & \mathbb{P}^3 \end{array}$$

where $\hat{X}_{2,2} \to \mathbb{P}^3$ is a blowup of 4 general coplanar points, and $\hat{X}_{2,2} \to X_{2,2}$ is a birational map that contracts the strict transform of the plane spanned by these four points.

(4) The $\mathrm{Aut}(X) = \mathbb{G}_m^2 \rtimes C_2$-action on $\mathbb{P}^2$ is generically free; any action of a finite subgroup is linearizable, by Proposition 2.1.

(5) Same as in Case (4), any action of a finite subgroup is linearizable.

4. Conic

We recall from [19] that the normal form of cubic threefolds X singular along a conic C is

$$x_1q_1(x_4,x_5)+x_2q_2(x_4,x_5)+x_3q_3(x_4,x_5)+ \\ +q_4(x_1,x_2,x_3)l(x_4,x_5)+c(x_4,x_5)=0,$$

where l is a linear form, q_1,q_2,q_3,q_4 are quadratic forms and c is a cubic form. Let $\Pi=\{x_4=x_5=0\}$. Then Π and C are $\mathrm{Aut}(X)$-invariant and

$$C=\{x_4=x_5=q_4=0\}\subset\Pi\subset X.$$

We may assume that C is smooth, otherwise the action either reduces to the case of a line, or is linearizable, via projection from the distinguished singularity. Similarly, we may assume that $\mathrm{Aut}(X)$ does not fix points in C.

Changing coordinates on $\mathbb{P}^4$, we may assume that $q_4=x_1x_2+x_3^2$ and $l=x_5$. Then

$$\{x_5=0\}\cap X=2\Pi+\Pi',$$

for some plane $\Pi'\subset X$. Note that $2\Pi+\Pi'$ is the only surface in the pencil of hyperplane sections containing Π that splits as a union of three planes. Hence, the plane Π' is also $\mathrm{Aut}(X)$-invariant. Apriori, we have two possibilities: $\Pi\neq\Pi'$ (general case) and $\Pi=\Pi'$ (special case).

In the general case, the line $\Pi\cap\Pi'$ intersects C in two distinct points. After another coordinate change, $\Pi'=\{x_3=x_5=0\}$ and X is given by

$$x_1(a_1x_4x_5+a_2x_5^2)+x_2(b_1x_4x_5+b_2x_5^2)+x_3(x_4^2+c_1x_4x_5+c_2x_5^2)+ \\ +(x_1x_2+x_3^2)x_5+e_1x_4^2x_5+e_2x_4x_5^2+e_3x_5^3=0,$$

for some $a_1,a_2,b_1,b_2,c_1,c_2,e_1,e_2,e_3\in k$. Now, changing coordinates $x_1\mapsto x_1+\alpha x_3+\beta x_4$ and $x_2\mapsto x_2+\gamma x_3+\delta x_4$, for some $\alpha,\beta,\gamma,\delta\in k$, we may further assume that $a_1=a_2=b_1=b_2=0$, and the defining equation of X simplifies to

$$x_3(x_4^2+c_1x_4x_5+c_2x_5^2)+(x_1x_2+x_3^2)x_5+e_1x_4^2x_5+e_2x_4x_5^2+e_3x_5^3=0.$$

Finally, changing coordinates $x_3\mapsto x_3+\epsilon x_5$ and $x_4\mapsto x_4+\varepsilon x_5$, for some $\epsilon,\varepsilon\in k$, we may assume that $c_1=c_2=0$, so that X is given by

$$x_3x_4^2+(x_1x_2+x_3^2)x_5+e_1x_4^2x_5+e_2x_4x_5^2+e_3x_5^3=0. \tag{4.1}$$

We may assume that $(e_1, e_2, e_3) \neq (0, 0, 0)$ in (4.1), since otherwise X would have additional singular point $[0:0:0:0:1]$, which would be fixed by $\mathrm{Aut}(X)$. Moreover, scaling coordinates x_1, x_2, x_3, x_4, x_5 as

$$x_1 \mapsto \frac{x_1}{s^2}, \quad x_2 \mapsto \frac{x_2}{s^2}, \quad x_3 \mapsto \frac{x_3}{s^2}, \quad x_4 \mapsto s x_4, \quad x_5 \mapsto s^4 x_5,$$

we scale (e_1, e_2, e_3) as $(s^6 e_1, s^9 e_2, s^{12} e_3)$, so we really have

$$[e_1 : e_2 : e_3] \in \mathbb{P}(6, 9, 12) \simeq \mathbb{P}(1, 3, 2).$$

Now, we turn to the special case when $\Pi = \Pi'$. Arguing as in the general case, we can change coordinates on $\mathbb{P}^4$ such that X is given by

$$x_5(x_1x_2 + x_3^2) + x_4^3 + e_1 x_4 x_5^2 + e_2 x_5^3 = 0, \tag{4.2}$$

for some $[e_1 : e_2] \in \mathbb{P}(4, 6)$. In this case, X has an A_2-singularity at a general point of the conic C.

In both cases, consider the unprojection of the cubic X from the plane Π, similar to the approach in [6, 5], for cubics with isolated singularities. Namely, if X is given by (4.1), we can introduce a new coordinate

$$y_6 = \frac{x_3x_4 + e_1x_4x_5 + e_2x_5^2}{x_5} = \frac{x_1x_2 + x_3^2 + e_3x_5^2}{-x_4},$$

which gives an $\mathrm{Aut}(X)$-equivariant birational map $X \dashrightarrow X_{2,2}$, where $X_{2,2}$ is a complete intersection of two quadrics in $\mathbb{P}^5_{y_1,\ldots,y_6}$ given by

$$y_3y_4 + e_1y_4y_5 + e_2y_5^2 - y_5y_6 = y_1y_2 + y_3^2 + e_3y_5^2 + y_4y_6 = 0.$$

In this case, $X_{2,2}$ has two isolated ordinary double points

$$[1:0:0:0:0:0], \quad [0:1:0:0:0:0],$$

for general $[e_1 : e_2 : e_3] \in \mathbb{P}(6, 9, 12)$. If X is given by (4.2), then we let

$$y_6 = \frac{x_4^2 + e_1x_5^2}{x_5} = \frac{x_1x_2 + x_3^2 + e_3x_5^2}{-x_4},$$

and obtain an $\mathrm{Aut}(X)$-equivariant birational map $X \dashrightarrow X_{2,2}$, where $X_{2,2}$ is given by

$$y_4^2 + e_1y_5^2 - y_5y_6 = y_1y_2 + y_3^2 + e_2y_5^2 + y_4y_6 = 0.$$

In this case, the singular locus

$$\mathrm{Sing}(X_{2,2}) = \{y_1y_2 + y_3^2 = y_4 = y_5 = y_6 = 0\}$$

is also a smooth conic, for general $[e_1 : e_2] \in \mathbb{P}(4, 6)$. Let

$$\beta : \tilde{\mathbb{P}} \to \mathbb{P}^4$$

be the blowup of Π. We have the commutative diagram

$$\begin{array}{ccc} & \tilde{\mathbb{P}} & \\ {}^{\beta}\swarrow & & \searrow^{\varpi} \\ \mathbb{P}^4 & \dashrightarrow & \mathbb{P}^1_{x_4,x_5} \end{array}$$

where ϖ is a $\mathbb{P}^3$-bundle and the dashed arrow is the projection from the plane Π. The restriction of ϖ to $\tilde{X}$, the strict transform of X, is a quadric surface bundle.

Proposition 4.1. *Let X be a cubic given by (4.1)*

$$x_3x_4^2 + (x_1x_2 + x_3^2)x_5 + e_1x_4^2x_5 + e_2x_4x_5^2 + e_3x_5^3 = 0,$$

with parameters $e_1, e_2, e_3 \in k$, such that $\mathrm{Aut}(X)$ does not fix any singular points of X. Then one of the following holds:

(1) *e_1, e_2, e_3 are general, and $\mathrm{Aut}(X) = \mathbb{G}_m \rtimes C_2$, generated by*

$$\tau_a : \mathrm{diag}(a, a^{-1}, 1, 1, 1),\ a \in k^\times, \quad \sigma_{(12)} : (\mathbf{x}) \mapsto (x_2, x_1, x_3, x_4, x_5).$$

(2) *$e_1, e_3 \neq 0$, $e_2 = 0$, and $\mathrm{Aut}(X) = C_2 \times (\mathbb{G}_m \rtimes C_2)$, generated by $\tau_a, \sigma_{(12)}$ and*

$$\eta_1 : \mathrm{diag}(1, 1, 1, -1, 1).$$

(3) *$e_1 = e_3 = 0$, $e_2 \neq 0$, and $\mathrm{Aut}(X) = C_3 \times (\mathbb{G}_m \rtimes C_2)$, generated by $\tau_a, \sigma_{(12)}$ and*

$$\eta_2 : \mathrm{diag}(1, 1, 1, \zeta_3, \zeta_3^2).$$

(4) *$e_1 = e_2 = 0$, $e_3 \neq 0$, and $\mathrm{Aut}(X) = C_4 \times (\mathbb{G}_m \rtimes C_2)$, generated by $\tau_a, \sigma_{(12)}$ and*

$$\eta_3 : \mathrm{diag}(1, 1, 1, \zeta_4, -1).$$

Proof. Note for general e_1, e_2, e_3, X has $\mathsf{A}_1 \times \mathbb{A}_1$-singularity at any point on the conic C different from the two points

$$[1 : 0 : 0 : 0 : 0], \quad [0 : 1 : 0 : 0 : 0] \in C,$$

so the image of G in $\mathrm{Aut}(C) = \mathsf{PGL}_2$ is contained in the infinite dihedral group $\mathbb{G}_m \rtimes C_2$. From the form of the equation, one sees that $\tau_a, \sigma \in \mathrm{Aut}(X)$ for all $a \in k^\times$ and $e_1, e_2, e_3 \in k$, and thus generate the full group $\mathbb{G}_m \rtimes C_2$. It follows that there is an exact sequence

$$0 \to H \to \mathrm{Aut}(X) \to \mathbb{G}_m \rtimes C_2 \to 0,$$

where H is the generic stabilizer of C. Then a computation of H based on the equation completes the proof. $\square$

Proposition 4.2. *Let X be a cubic given by (4.2)*

$$x_5(x_1x_2+x_3^2)+x_4^3+e_1x_4x_5^2+e_2x_5^3=0,$$

with parameters $e_1, e_2 \in k$ such that $\mathrm{Aut}(X)$ does not fix any singular points of X. Then one of the following holds:

(5) *$e_1, e_2 \neq 0$, and $\mathrm{Aut}(X) = C_2 \times \mathsf{PGL}_2$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{PGL}_2$ acts via*

$$\begin{pmatrix} a^2 & b^2 & \zeta_4 ab & & \\ c^2 & d^2 & \zeta_4 cd & & \\ \frac{2ac}{\zeta_4} & \frac{2bd}{\zeta_4} & ad+bc & & \\ & & & ad-bc & \\ & & & & ad-bc \end{pmatrix};$$

and C_2 acts via

$$\eta_1 : \mathrm{diag}(1,1,1,-1,-1).$$

(6) *$e_1 \neq 0$, $e_2 = 0$, and $\mathrm{Aut}(X) = C_4 \times \mathsf{PGL}_2$, generated by the PGL_2-action described in case (5) and*

$$\eta_2 : \mathrm{diag}(1,1,1,\zeta_4,-\zeta_4).$$

(7) *$e_1 = 0, e_2 \neq 0$, and $\mathrm{Aut}(X) = C_6 \times \mathsf{PGL}_2$, generated by the PGL_2-action described in case (5) and*

$$\eta_3 : \mathrm{diag}(1,1,1,-\zeta_3,-1)$$

Proof. In this case, we observe that $\mathsf{PGL}_2 \subset \mathrm{Aut}(X)$ with the generators given above. As before, one can directly compute the generic stabilizer H of C to obtain the three cases in the assertions. Note that H always commutes with PGL_2. □

We turn to the problem of linearization.

Theorem 4.3. *Let X be the cubic given by (4.1) or (4.2) and $G = \mathfrak{D}_n$ the dihedral group generated by τ_a and $\sigma_{(12)}$ with $a = \zeta_n$, for some even n. Then the G-action on X is not linearizable. In particular, the $\mathrm{Aut}(X)$-action on X is not linearizable.*

Proof. The group G contains the Klein four-group $\langle \tau_{-1}, \sigma_{(12)} \rangle$ where σ_{12} fixes a cubic surface and the residual C_2 acting on S fixes a smooth elliptic curve. The assertion then follows from [6, Proposition 2.6]. □

Remark 4.4. Forms over non-closed fields of cubics given by (4.2) have been considered in [9, Section 11.3]. In the birational models as intersections of two quadrics in $\mathbb{P}^5$, the singular locus is a conic without rational points. The rationality of such threefolds remains an open problem.

On the other hand, in the equivariant context, for the action of $G = C_2^2$, the corresponding conic has no G-fixed points; and the G-action is not linearizable, by Theorem 4.3.

Theorem 4.5. *Let X be a cubic given by (4.1) or (4.2). Then X is G-unirational for all finite $G \subseteq \mathrm{Aut}(X)$.*

Proof. The plane $\Pi \subset X$ (spanned by the conic C) is necessarily G-invariant. The assertion then follows from Proposition 2.3. ☐

5. The chordal cubic

There is a unique cubic threefold $X \subset \mathbb{P}^4$ such that $\mathrm{Sing}(X)$ is a rational normal quartic curve. It is known as the *chordal cubic*, and is given by

$$X = \{x_1x_4^2 + x_2^2x_5 - x_1x_3x_5 - 2x_2x_3x_4 + x_3^3 = 0\} \subset \mathbb{P}^4_{x_1,\ldots,x_5}. \tag{5.1}$$

This X is the secant variety of its singular locus $C := \mathrm{Sing}(X)$. We have

$$\mathrm{Aut}(X) = \mathsf{PGL}_2,$$

where PGL_2 acts on $\mathbb{P}^4$ via the usual embedding $C = \mathbb{P}^1 \hookrightarrow \mathbb{P}^4$. The linear system $|\mathcal{O}_X(2) - C|$ gives rise to a PGL_2-equivariant rational map $\rho : X \dashrightarrow S$, where $S = \mathbb{P}^2$ is the Veronese surface in $\mathbb{P}^5$. This map fits into the following PGL_2-equivariant commutative diagram:

$$\begin{array}{ccccc} & & \tilde{X} & & \\ & \swarrow^{\beta} & & \searrow^{\varpi} & \\ X & & \overset{\rho}{\dashrightarrow} & & S \end{array}$$

where β is the blowup of C in X and ϖ is a $\mathbb{P}^1$-bundle.

Let $G \subset \mathrm{Aut}(X)$ be a finite subgroup. Then G is one of the following

$$C_n, \quad \mathfrak{D}_n, \quad \mathfrak{A}_4, \quad \mathfrak{S}_4, \quad \text{and} \quad \mathfrak{A}_5.$$

For all such G, the induced action on S is generically free, and linearizable. Moreover, the G-action on X satisfies Condition **(A)**, indeed, the Klein four-group $C_2^2 \subset \mathsf{PGL}_2$, acting via

$$\mathrm{diag}(1, -1, 1, -1, 1) \quad \text{and} \quad (\mathbf{x}) \mapsto (x_5, x_4, x_3, x_2, x_1),$$

fixes the smooth point $[1:0:1:0:1] \in X$, and all other abelian groups are cyclic. Applying Proposition 2.1, we obtain linearizability of the G-action on X. To summarize, we obtain

Theorem 5.1. *Let X be the chordal cubic threefold given by (5.1), and $G \subset \mathrm{Aut}(X) = \mathsf{PGL}_2$ a finite subgroup. Then the G-action on X is linearizable.*

Remark 5.2. We note that although all finite subgroups of $\mathrm{Aut}(X)$ in this case are linearizable, the $\mathrm{Aut}(X)$-action on X is not linearizable. Indeed, from [14, Section 2], we see that the $\mathbb{P}^1$-bundle $\varpi : \tilde{X} \to \mathbb{P}^2$ is the Schwarzenberger bundle $\mathcal{S}_3$, in the notation of [2, Definition 1.2.7]. Precisely, it is the projectivization of the pullback of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,4)$ under the double cover $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ ramified along a conic. The connected component of the identity $\mathrm{Aut}^\circ(\tilde{X}) = \mathsf{PGL}_2$. By [3, Theorem F], there is no $\mathrm{Aut}^\circ(\tilde{X})$-equivariant birational map to $\mathbb{P}^3$.

6. Plane

By [19, Proposition 4.2], a cubic threefold X singular along the plane

$$\Pi = \{x_4 = x_5 = 0\} \subset \mathbb{P}^4$$

is given by

$$x_1q_1(x_4, x_5) + x_2q_2(x_4, x_5) + x_3q_3(x_4, x_5) + c(x_4, x_5) = 0,$$

where q_1, q_2, q_3 are quadratic forms and c is a cubic form in x_4, x_5. Viewing the first three terms as a quadratic form in x_4, x_5, over $k(x_1, x_2, x_3)$, we find that its discriminant defines a conic $C \subset \Pi = \mathbb{P}^2_{x_1,x_2,x_3}$. If C is non-reduced or singular then there is a distinguished point on Π fixed by any group action, implying the linearizability. Thus, we may assume that C is smooth.

Proposition 6.1. *Let X be a cubic threefold singular along a plane. Assume that $\mathrm{Aut}(X)$ does not fix any singular points of X. Then, up to isomorphism, X is given by*

$$x_1x_4^2 + x_2x_4x_5 + x_3x_5^2 = 0, \tag{6.1}$$

and $\mathrm{Aut}(X)$ is generated by elements

$$(\mathbf{x}) \mapsto (\mathbf{x}) \cdot \begin{pmatrix} d^2 & -2bd & b^2 & & \\ -cd & (ad+bc) & -ab & & \\ c^2 & -2ac & a^2 & & \\ & & & a & c \\ & & & b & d \end{pmatrix}, \quad ad - bc \neq 0, \tag{6.2}$$

and

$$(6.3)\qquad (\mathbf{x}) \mapsto (\mathbf{x}) \cdot \begin{pmatrix} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & -\alpha & -\beta & \gamma & 0 \\ \alpha & \beta & 0 & 0 & \gamma \end{pmatrix}, \quad \alpha, \beta \in k, \gamma \in k^{\times}.$$

Proof. Recall that $\mathrm{Aut}(X)$ preserves the conic $C \subset \Pi$. Up to isomorphism, we may assume that

$$C = \{4x_1x_3 - x_2^2 = x_4 = x_5 = 0\}.$$

Since X does not contain a distinguished singular point on Π, we may change variables to obtain the unique equation (6.1) for X.

Since $\mathrm{Aut}(X)$ preserves C, its effective action on Π is a subgroup of PGL_2. We note that $\mathrm{Aut}(X)$ contains a subgroup isomorphic to

$$\mathsf{GL}_2 = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad ad - bc \neq 0 \right\},$$

generated by elements of the form (6.2), which acts via PGL_2 on Π. It follows that there is an exact sequence

$$1 \to H \to \mathrm{Aut}(X) \to \mathsf{PGL}_2 \to 1,$$

where H is the generic stabilizer of Π. A direct computation shows that H is generated by elements of the form (6.3).

□

Proposition 6.1 implies that $\mathrm{Aut}(X)$ is isomorphic to the subgroup in GL_3 consisting of 3×3 matrices

$$\begin{pmatrix} a & c & 0 \\ b & d & 0 \\ \alpha & \beta & 1 \end{pmatrix}.$$

This can also be explained as follows. Let $f\colon \tilde{X} \to X$ be the blowup of Π, and let E be the preimage of Π in $\tilde{X}$. Then

$$(6.4)\qquad \tilde{X} = \mathbb{P}^1_{y_1,y_2} \times \mathbb{P}^2_{z_1,z_2,z_3}.$$

We may assume that E is given by $z_3 = 0$, so that we can identify $E = \mathbb{P}^1_{y_1,y_2} \times \mathbb{P}^1_{z_1,z_2}$. Then

$$\mathrm{Aut}(X) = \mathrm{Aut}(\tilde{X}, E),$$

where the latter group consists of automorphisms

$$(y_1, y_2) \times (z_1, z_2, z_3) \mapsto \\ (ay_1 + by_2, cy_1 + dy_2) \times (az_1 + bz_2 + \alpha z_3, cz_1 + dz_2 + \beta z_3, z_3).$$

The birational morphism f is equivariant with respect to the described actions of $\mathrm{Aut}(\tilde{X}, E)$, and can be explicitly presented as follows. Consider the Segre embedding $\mathbb{P}^1_{y_1,y_2} \times \mathbb{P}^2_{z_1,z_2,z_3} \hookrightarrow \mathbb{P}^5$ given by

$$(y_1, y_2) \times (z_1, z_2, z_3) \mapsto (y_1z_1, y_1z_2, y_1z_3, y_2z_1, y_2z_2, y_2z_3).$$

Composing it with the linear projection $\mathbb{P}^5_{t_1,t_2,t_3,t_4,t_5,t_6} \dashrightarrow \mathbb{P}^4$ given by

$$(\mathbf{t}) \mapsto (t_1, -t_2 - t_4, t_5, t_6, t_3), \tag{6.5}$$

we obtain the map f; the resulting image of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^4$ is the cubic X given by (6.1). The map (6.5) is a projection from the point

$$[0 : 1 : 0 : -1 : 0 : 0] \in \mathbb{P}^5.$$

This point is not in the image of $\mathbb{P}^1 \times \mathbb{P}^2$, but it is fixed by the image of $\mathrm{Aut}(\tilde{X}, E)$ in PGL_6. Note that f induces a double cover $E \to \Pi$ which is ramified in C, which explains why this conic is $\mathrm{Aut}(X)$-invariant.

Theorem 6.2. *Let X be the cubic threefold given by (6.1) and $G \subset \mathrm{Aut}(X)$ a finite group. Then the G-action on X is linearizable.*

Proof. We see that the $\mathrm{Aut}(X)$-equivariant birational model $\tilde{X}$ of X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$, with coordinates as in (6.4). Any finite subgroup $G \subset \mathrm{Aut}(X)$ acts linearly and generically freely on $\mathbb{P}^2_{z_1,z_2,x_3}$, and lifts to the vector bundle

$$\mathbb{A}^2_{y_1,y_2} \times \mathbb{P}^2_{z_1,z_2,x_3} \to \mathbb{P}^2_{z_1,z_2,x_3}.$$

By Proposition 2.1, the G-action is linearizable. □

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