COMPUTING THE EQUIVARIANT BRAUER GROUP

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Abstract. Let X be a smooth projective rational variety carrying a regular action of a finite abelian group G. We give examples of effective computation of the Brauer group of the quotient stack $[X/G]$ in dimensions 2 and 3 using residues in Galois cohomology and the geometry of fixed loci. In particular, we compute $\mathrm{Br}([X/G])$ for all G-minimal del Pezzo surfaces.

1. Introduction

Let k be a field of characteristic 0. Consider a smooth projective rational variety X over an algebraic closure $\bar{k}$ of k carrying a regular and generically free action of a finite group G. Studying such actions up to equivariant birationality is a classical and active area in birational geometry. Of particular interest is the *linearizability problem*, which asks whether or not the G-action on X is *linearizable*, i.e., equivariantly birational to a linear G-action on $\mathbb{P}^n$. An arithmetic counterpart of this problem is the classical *rationality problem* over nonclosed fields k, where the Galois action is considered as an analogue of the G-action.

An established strategy to study both linearizability and rationality problems is to seek nontrivial birational invariants. The similarities between the two problems are well reflected in the following invariant: the group cohomology

$$\mathrm{H}^1(H, \mathrm{Pic}(X_{\bar{k}})), \quad H \subseteq G. \tag{1.1}$$

This invariant was first studied by Yu. Manin [13] in the arithmetic setting, where $G = \mathrm{Gal}(\bar{k}/k)$ is the Galois group and the action is induced from the Galois action on $\bar{k}$. F. Bogomolov and Y. Prokhorov [4] extended it to the equivariant setting, when G is a finite group and the action comes from geometric automorphisms of $X_{\bar{k}}$. In the respective cases, the vanishing of (1.1) for every subgroup H of G is a necessary condition for X to be stably rational over k, and for the G-action on $X_{\bar{k}}$ to be stably linearizable. Applications of this invariant to the study of the linearizability problem can be found in [6, 8].

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In the arithmetic setting, when k is a nonclosed field, the Leray spectral sequence for the Galois action gives rise to a well-known long exact sequence

$$(1.2)\quad 0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)} \to \mathrm{Br}(k) \to \\ \to \ker(\mathrm{Br}(X) \to \mathrm{Br}(X_{\bar{k}})) \to \mathrm{H}^1(\mathrm{Gal}(\bar{k}/k), \mathrm{Pic}(X_{\bar{k}})) \to \mathrm{H}^3(\mathrm{Gal}(\bar{k}/k), \bar{k}^\times),$$

where $\mathrm{Br}(k)$ and $\mathrm{Br}(X)$ are the Brauer group of k and X respectively. The group $\ker(\mathrm{Br}(X) \to \mathrm{Br}(X_{\bar{k}}))$ is known as the *algebraic part* of the Brauer group.

In the equivariant setting, when $k = \bar{k}$, the Leray spectral sequence for the G-action produces an exact sequence similar to (1.2)

$$(1.3)\quad 0 \to \mathrm{Hom}(G, k^\times) \to \mathrm{Pic}(X, G) \to \mathrm{Pic}(X)^G \to \mathrm{H}^2(G, k^\times) \to \\ \to \mathrm{Br}([X/G]) \to \mathrm{H}^1(G, \mathrm{Pic}(X)) \to \mathrm{H}^3(G, k^\times),$$

where $\mathrm{Pic}(X, G)$ is the group of G-linearizable line bundles on X and $\mathrm{Br}([X/G])$ is the Brauer group of the quotient stack $[X/G]$. The group $\mathrm{Br}([X/G])$ is a G-stably birational invariant, and can be viewed as an analogue of the algebraic part of the Brauer group as in (1.2).

From now on, we focus on the equivariant setting with $k = \bar{k}$, and study the groups $\mathrm{Br}([X/G])$ and $\mathrm{H}^1(G, \mathrm{Pic}(X))$. Given a G-action on X, it can be computationally challenging to find the induced G-action on $\mathrm{Pic}(X)$, as this requires a thorough analysis of divisors on X. On the other hand, the geometry of the fixed locus X^G contains rich information readily available in the equivariant setting, but absent in the arithmetic setting (where the Galois fixed locus simply consists of all k-rational points).

When G is a cyclic group acting on a smooth rational surface X with *maximal* stabilizers, the works of F. Bogomolov and Y. Prokhorov, and E. Shinder [4, 14] give a formula for $\mathrm{H}^1(G, \mathrm{Pic}(X))$ only involving information about the G-fixed curves on X. Generalizing this, for any finite group G acting on a smooth projective variety X, A. Kresch and Y. Tschinkel gave an algorithm to compute $\mathrm{Br}([X/G])$ which only requires information about divisors with nontrivial stabilizers, and presented examples of effective computations when X is a rational surface [11, 12].

In this note, we extend the scope of applications of this algorithm to dimension 3. In particular, we produce a nontrivial class in $\mathrm{Br}([\tilde{X}/G])$ for a smooth model $\tilde{X}$ of a singular cubic threefold X, and showcase the connection between $\mathrm{Br}([\tilde{X}/G])$ and $\mathrm{H}^1(G, \mathrm{Pic}(\tilde{X}))$ through this example (see Remark 3.2). We also complete the computations of $\mathrm{Br}([X/G])$ in dimension 2 for all G-minimal del Pezzo surfaces X and finite abelian groups G.

Here is a road map of the paper. In Section 2, we review basic facts about the Brauer group of the quotient stack. In Section 3, we produce an example with nontrivial $\mathrm{Br}([X/G])$ in dimension 3. We compute $\mathrm{Br}([X/G])$ in dimension 2 in Section 4.

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2. Preliminaries

2.1. **Brauer groups.** Let X be a smooth projective variety over a field k and n a positive integer invertible in k. Let $\mathrm{H}^i(k(X), \mu_n)$ be the Galois cohomology of the function field of X with μ_n-coefficients, where μ_n is the étale k-group scheme of the n^{th}-roots of unity (in particular, $\mu_n \simeq \mathbb{Z}/n\mathbb{Z}$ when k is algebraically closed).

If v is a discrete valuation on the field $k(X)$, one has the residue maps

$$\partial_v^i : \mathrm{H}^i(k(X), \mu_n^{\otimes j}) \to \mathrm{H}^{i-1}(\kappa(v), \mu_n^{\otimes(j-1)}),$$

where $\kappa(v)$ is the residue field. In particular, for $a \in \mathrm{H}^1(k(X), \mathbb{Z}/2\mathbb{Z})$ and for $(a, b) \in \mathrm{H}^2(k(X), \mathbb{Z}/2\mathbb{Z})$, one has

$$\partial_v^1(a) = v(a) \bmod 2 \in \mathbb{Z}/2\mathbb{Z},$$
$$\partial_v^2(a,b) = (-1)^{v(a)v(b)}\overline{a^{v(b)}b^{-v(a)}} \in \kappa(v)^\times/(\kappa(v)^\times)^2, \tag{2.1}$$

where for a unit u in the valuation ring of v, we denote by $\bar{u}$ its image in the residue field $\kappa(v)$.

For an irreducible divisor D on X, we denote by v_D the associated divisorial valuation on $k(X)$, ∂_D^i the corresponding residue maps, and $\kappa(v_D)$ or $\kappa(D)$ the residue field. Similarly, for $\xi \in X^{(1)}$ a codimension 1 point of X, we denote by v_ξ and ∂_ξ^i the associated valuation and residue maps. The n-torsion in the Brauer group of X can be computed via

$$\mathrm{Br}(X)[n] = \bigcap_D \ker(\partial_D^2), \tag{2.2}$$

where D runs over all irreducible divisors on X (see [5, Proposition 4.2.3] and [5, Theorem 4.1.1]; note that we assume that X is smooth and projective).

Now let k be an algebraically closed field of characteristic zero and X a smooth projective variety over k carrying a generically free regular action of a finite group G. An analogue of the classical formula (2.2) for $\mathrm{Br}([X/G])$, the Brauer group of the quotient stack $[X/G]$, is established in [11, 12]:

Proposition 2.1. *Let k be an algebraically closed field of characteristic zero and X a smooth projective variety over k carrying a generically free regular action of a finite group G. For any irreducible divisor D on X, we denote by I_D the stabilizer group*

$$I_D := \{g \in G \mid g \text{ acts trivially on } D\}.$$

Then the n-torsion subgroup of $\mathrm{Br}([X/G])$, denoted by $\mathrm{Br}([X/G])[n]$, can be computed via

$$\mathrm{Br}([X/G])[n] = \bigcap_D \ker\left(|I_D| \cdot \partial^2_{D'}\right) \subset H^2(k(X)^G, \mathbb{Z}/n\mathbb{Z}), \tag{2.3}$$

where D runs over all irreducible divisors on X, $|I_D|$ is the order of I_D, and $\partial^2_{D'}$ is the residue map in degree 2 corresponding to the divisorial valuation on $k(X)^G$ given by the image D' of D.

Proof. See [11, Proposition 4.2], [12, Section 4], and [10, Proposition 2.2]. □

2.2. **Rational surfaces.** In this subsection, we review an effective algorithm provided in [11] to compute $\mathrm{Br}([X/G])$ when X is a rational surface.

Let X be a smooth projective rational surface carrying a generically free regular action of a finite group G. Recall that the G-action on X is called *in standard form* if there exists a G-invariant simple normal crossing divisor $\mathcal{D} \subset X$ such that

- the G-action on $X \setminus \mathcal{D}$ is free, and
- for any $g \in G$ and any irreducible component D of $\mathcal{D}$, either $g(D) = D$ or $g(D) \cap D = \emptyset$.

Any G-action on X can be brought into standard form via successive blowups [15, Theorem 3.2]. The following proposition provides an effective algorithm only involving divisors with nontrivial stabilizers to determine $\mathrm{Br}([X/G])$.

Proposition 2.2 ([11, Proposition 4.2, Corollary 4.6])**.** *Let X be a smooth projective rational surface carrying a finite group G-action in standard form. Then the group $\mathrm{Br}([X/G])[n]$ can be identified with the kernel of the map*

$$\bigoplus_{[\xi]\in X^{(1)}/G} \mathrm{H}^1(\mathrm{Spec}(k(\xi)^{D_\xi}), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\oplus\partial^1} \bigoplus_{[\mathfrak{p}]\in X/G} \mathbb{Z}/n\mathbb{Z}, \tag{2.4}$$

where the sum on the left runs over G-orbit representatives $[\xi]$ of codimension 1 points on X such that the stabilizer group I_ξ has cardinality n,

$$D_\xi := \{g \in G \mid \xi \cdot g = \xi\},$$

and the sum on the right runs over G-orbit representatives $[\mathfrak{p}]$ of points of X.

Using Proposition 2.2, we compute $\mathrm{Br}([X/G])$ for all G-minimal del Pezzo surfaces with abelian groups G in Section 4. Here we introduce the notation and demonstrate the process in detail with a concrete example.

Example 2.3. We consider the case $\boxed{3.36}$ in [1]. Let X be a smooth cubic surface given by

$$\{w^3 + x^3 + xy^2 + z^3 = 0\} \subset \mathbb{P}^3_{w,x,y,z},$$

with an action of $G = C_3 \times C_6$ generated by

$$\sigma\colon (w, x, y, z) \mapsto (\zeta_3 w, x, y, z),$$
$$\tau\colon (w, x, y, z) \mapsto (w, x, -y, \zeta_3 z).$$

We list the stratification of curves with nontrivial stabilizers

i	Curve ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$
1	$\{y = 0\} \cap X$	$C_2 = \langle \tau^3 \rangle$	G	1	0
2	$\{w = 0\} \cap X$	$C_3 = \langle \sigma \rangle$	G	1	0
3	$\{z = 0\} \cap X$	$C_3 = \langle \tau^2 \rangle$	G	1	0

where

- the second column displays equations of the curves ξ_i with nontrivial generic stabilizers;
- the third column displays the stabilizers I_{ξ_i} of ξ_i;
- the fourth column displays the groups $D_{\xi_i} = \{g \in G \mid \xi_i \cdot g = \xi_i\}$;
- the fifth column displays the genera of ξ_i;
- the sixth column displays the genera of ξ_i/D_{ξ_i}, computed via the Riemann-Hurwitz formula.

One can check that the G-action on X is in standard form: the G-action is free in the complement of the simple normal crossing divisor $\xi_1 \cup \xi_2 \cup \xi_3$ in X, and G leaves invariant each of the curves ξ_i for $i = 1, 2, 3$. By Proposition 2.2, we know that $\mathrm{Br}([X/G])[n]$ is possibly nonzero only when $n = 2$ or 3.

For $n = 3$, observe that $\xi_2 \cap \xi_3$ consists of three points where two of them are in the same G-orbit. It follows that the quotient curves ξ_2/G and ξ_3/G are both rational and meet at two points, denoted by p_1 and p_2. The kernel of (2.4) consists of classes of functions ramified only at p_1 and p_2, where the ramification indices are distinct at two points. It follows that

$$\mathrm{Br}([X/G])[3] = \mathbb{Z}/3.$$

For $n=2$, the quotient curve ξ_1/G has genus 0. Then

$$\ker\left(\mathrm{H}^1(\mathrm{Spec}(k(\xi_1))^G, \mathbb{Z}/2\mathbb{Z}) \to \bigoplus_{[\mathfrak{p}]\in X/G} \mathbb{Z}/2\mathbb{Z}\right) = 0$$

(see [5, Proposition 4.2.1(b)]). Combining the 2-torsion and 3-torsion subgroups, we conclude that

$$\mathrm{Br}([X/G]) = \mathbb{Z}/3\mathbb{Z}.$$

3. A cubic threefold

Let k be an algebraically closed field of characteristic zero, and X the cubic threefold given by

$$X = \{F = 0\} \subset \mathbb{P}^4_{y_1,\dots,y_5},$$

where

$$F = y_3^2(y_1 - y_4) + y_5^2(y_1 + y_2) - 2y_1y_3y_5 + f,$$
$$f = -y_1^2y_2 - y_1y_2^2 + y_1^2y_4 - y_1y_4^2 + y_2^2y_4 - y_2y_4^2 - 2y_1y_2y_4.$$

Let $G = C_2$ act on X by

$$(y_1, y_2, y_3, y_4, y_5) \mapsto (y_1, y_2, -y_3, y_4, -y_5).$$

The singular locus of X consists of six ordinary double points:

$$p_1 = [0:0:0:1:-1], \quad p_2 = [0:0:0:1:1], \quad p_3 = [1:0:-1:0:-1],$$

$$p_4 = [1:0:1:0:1], \quad p_5 = [0:1:-1:0:0], \quad p_6 = [0:1:1:0:0].$$

Let $\tilde{X}$ be the blowup of X at $p_1,\dots,p_6$ and the G-invariant curve given by

$$\{y_3 = y_5 = 0\} \cap X.$$

Since $p_1,\dots,p_6$ are ordinary double points, $\tilde{X}$ is smooth.

Proposition 3.1. *In the above notation, the group* $\mathrm{Br}([\tilde{X}/G])$ *is nonzero. More precisely, the class*

$$\alpha = \left(\frac{f}{(y_2-y_4)^3}, \frac{-y_1y_4 + y_1y_2 - y_2y_4}{(y_2-y_4)^2}\right) \in \mathrm{H}^2(k(X)^G, \mathbb{Z}/2\mathbb{Z})$$

is nontrivial, and belongs to $\mathrm{Br}([\tilde{X}/G])$*. In particular, the G-action on X is not linearizable.*

The rest of this section is devoted to the proof of Proposition 3.1. First, let

$$q = -y_1y_4 + y_1y_2 - y_2y_4$$

and

$$\alpha = (a, b) \in \mathrm{H}^2(k(X)^G, \mathbb{Z}/2\mathbb{Z}), \tag{3.1}$$

where

$$a := \frac{f}{(y_2 - y_4)^3}, \quad b := \frac{q}{(y_2 - y_4)^2}.$$

Since $\tilde{X}$ is smooth and projective, by Proposition 2.1, it suffices to check that for all divisorial valuations on $k(\tilde{X})^G$ given by the image D' of a divisor D on $\tilde{X}$, we have

$$|I_D| \cdot \partial^2_{D'}(\alpha) = 0. \tag{3.2}$$

Note that no singular point of X lies on the plane section of X given by $y_3 = y_5 = 0$. In particular, we may blow up the singular points and the curve $\{y_3 = y_5 = 0\} \cap X$ independently.

Blowup of $y_3 = y_5 = 0$. The blowup Y of X along the cubic curve

$$\{y_3 = y_5 = 0\} \cap X$$

is given by

$$Y = \{F = y_3z_5 - y_5z_3 = 0\} \subset \mathbb{P}^4_{y_1,\dots,y_5} \times \mathbb{P}^1_{z_3,z_5}.$$

We first compute residues in the affine chart U of Y given by $y_4 = z_3 = 1$. Put

$$g = y_1 - y_4 + y_1z_5^2 + y_2z_5^2 - 2y_1z_5.$$

Then $y_5 = y_3z_5$ and the equation of U is

$$U : y_3^2\bar{g} + \bar{f} = 0,$$

where $\bar{g}$ (resp. $\bar{f}$) is the affine equation of g (resp. f) in the chart $y_4 = 1$. The action of G on U is

$$(y_1, y_2, y_3, z_5) \mapsto (y_1, y_2, -y_3, z_5).$$

Then U/G is the affine rational variety

$$U/G \subset \mathbb{A}^4_{y_1,y_2,w_3,z_5}, \quad w_3\bar{g} + \bar{f} = 0. \tag{3.3}$$

In $k(U)^G$, one has $\bar{f} = -w_3\bar{g}$, and we rewrite

$$\alpha = \left(\frac{-w_3\bar{g}}{(y_2 - 1)^3}, \frac{-y_1 + y_1y_2 - y_2}{(y_2 - 1)^2}\right) \in \mathrm{H}^2(k(U)^G, \mathbb{Z}/2) = \mathrm{H}^2(k(X)^G, \mathbb{Z}/2).$$

We then compute residues of α at divisors of $U/G \simeq \mathbb{A}^3_{y_1,y_2,z_5}$. From the definition of α, we know that $\partial_{D'}(\alpha) = 0$ for any divisor D' of U/G except possibly when D' is one of the following four divisors:

$$D'_1 : w_3 = 0, \quad D'_2 : \bar{g} = 0, \quad D'_3 : y_2 - 1 = 0, \quad D'_4 : -y_1 + y_1y_2 - y_2 = 0.$$

We consider these four cases:

(1) D'_1 is the image of the divisor $D_1 : y_3 = 0$ on U, with $|I_{D_1}| = 2$. Hence condition (3.2) is satisfied for D_1. We claim that

$$\partial_{D'_1}(\alpha) = q = -y_1 + y_1y_2 - y_2 \neq 0 \text{ in } \kappa(D'_1)^\times/(\kappa(D'_1)^\times)^2. \tag{3.4}$$

Indeed, at the generic point of D'_1, the function $\bar{g}$ is invertible. Hence $\kappa(D'_1)$ is the field of functions of the subscheme

$$\{\bar{f} = 0\} \subset \mathbb{A}^3_{y_1,y_2,z_5},$$

which is a purely transcendental extension of the function field of the cubic curve

$$C : \bar{f} = -y_1^2y_2 - y_1y_2^2 + y_1^2 - y_1 + y_2^2 - y_2 - 2y_1y_2 = 0 \subset \mathbb{A}^2_{y_1,y_2}.$$

Since the function $y_2 - 1$ is invertible at the generic point of C, we may write:

$$\bar{f}(y_1, y_2) = \bar{f}\left(\frac{q + y_2}{y_2 - 1}, y_2\right) = \frac{y_2^2(-q-5) - 4qy_2 - q^2 - q}{y_2 - 1}.$$

The discriminant of

$$y_2^2(-q-5) - 4qy_2 - q^2 - q$$

as a quadratic polynomial in the variable y_2 over $k(q)$ is

$$d = q(-4q^2 - 8q - 20)$$

with $-4q^2 - 8q - 20$ being a nonsquare in $k(q)$. We obtain:

$$\kappa(D_1) = k(q)(\sqrt{d})(z_5).$$

Hence the kernel of the natural map

$$k(q)^\times/(k(q)^\times)^2 \to \kappa(D'_1)^\times/(\kappa(D'_1)^\times)^2$$

is generated by d, so that q is a nonzero element in $\kappa(D'_1)^\times/(\kappa(D'_1)^\times)^2$.

(2) $\partial_{D'_2}(\alpha)$ is the image of $-y_1 + y_1y_2 - y_2$ in $\kappa(D'_2)^\times/(\kappa(D'_2)^\times)^2$. By the definition of D'_2, we have a relation

$$y_1 - 1 + y_1z_5^2 + y_2z_5^2 - 2y_1z_5 = 0$$

in $\kappa(D_2')$, where we still write y_1, y_2, z_5 for their images in $\kappa(D_2')^\times$. We rewrite this condition as:

$$(y_1+y_2)(z_5-\frac{y_1}{y_1+y_2})^2+\frac{-y_1+y_1y_2-y_2}{y_1+y_2}=0,$$

so that

$$-y_1+y_1y_2-y_2=-(y_1+y_2)^2(z_5-\frac{y_1}{y_1+y_2})^2$$

is a square in $\kappa(D_2')^\times$ and $\partial_{D_2'}(\alpha)=0$.

(3) $\partial_{D_3'}(\alpha)=(-y_1+y_1y_2-y_2)^3|_{y_2=1}=-1$ is a square in $\kappa(D_3')^\times$. So we have that $\partial_{D_3'}(\alpha)=0$.

(4) $\partial_{D_4'}(\alpha)$ is the image of $\frac{\bar{f}}{(y_2-1)^3}$ in $\kappa(D_4')^\times/(\kappa(D_4')^\times)^2$. In the field $\kappa(D_4')$, we have a relation

$$y_1=\frac{y_2}{y_2-1}.$$

Then we find that in $\kappa(D_4')$,

$$\frac{\bar{f}}{(y_2-1)^3}=-\frac{5y_2^2}{(y_2-1)^4}$$

is a square, hence $\partial_{D_4'}(\alpha)=0$.

The computations in the remaining charts $y_1=z_3=1$, $y_2=z_3=1$, and $y_i=z_5=1, i=1,2,4$ of Y are similar. Hence we have verified that the condition (3.2) holds for all divisors on $\tilde{X}/G$ except the images of the exceptional divisors of the blowups of six singular points.

Exceptional divisors of $\mathrm{Bl}_{p_1,p_2}(X)$**.** Let v be the valuation on $k(X)^G$ corresponding to the exceptional divisor of the blowup of X at the G-orbit of two singular points p_1 and p_2. We work with the affine chart $y_4=1$. Put

$$g_1=y_1,\quad g_2=y_2,\quad g_3=y_3,\quad g_4=y_5^2-1,$$

one may view the union of p_1 and p_2 as a variety given by

$$\{g_1=g_2=g_3=g_4=0\}\subset\mathbb{A}^4_{y_1,y_2,y_3,y_5}.$$

The blowup $\mathrm{Bl}_{p_1,p_2}(\mathbb{A}^4)$ is given by

$$\{g_iz_j-g_jz_i=0\mid i,j=1,\ldots,4\}\subset\mathbb{A}^4_{y_1,y_2,y_3,y_5}\times\mathbb{P}^3_{z_1,z_2,z_3,z_4}.$$

The induced G-action is given by

$$(y_1,y_2,y_3,y_5)\times(z_1,z_2,z_3,z_4)\mapsto(y_1,y_2,-y_3,-y_5)\times(z_1,z_2,-z_3,z_4).$$

In the affine chart $z_4 = 1$, the defining equations are equivalent to the change of variables

$$y_i = z_i(y_5^2 - 1), \quad i = 1, 2, 3. \tag{3.5}$$

The exceptional divisor E is given by $y_5^2 = 1$. Note that it consists of two components in the same G-orbit. So it gives rise to a divisorial valuation v of $k(X)^G$. Since $a, b \in k(X)^G$, after substituting (3.5) into (3.1), one can compute

$$v\left(\frac{f}{(y_2 - y_4)^3}\right) = v\left(\frac{q}{(y_2 - y_4)^2}\right) = 1. \tag{3.6}$$

From (2.1), one has

$$\partial_v(\alpha) = \frac{f}{q(z_2(y_5^2 - 1) - 1)} = -1 \in \kappa(v)^\times/(\kappa(v)^\times)^2$$

where the second equality is obtained via evaluation at $y_5^2 = 1$. It follows from (3.6) that $\partial_v(a) = 0$.

The computations of the residue of α along exceptional divisors of blowups of the other two G-orbits of singular points are similar. We summarize them below.

Exceptional divisors of $\mathrm{Bl}_{p_3,p_4}(X)$**.** Let v be the valuation on $k(X)^G$ corresponding to the exceptional divisors of $\mathrm{Bl}_{p_3,p_4}(X)$. Similarly as (3.5), after choosing an appropriate affine chart and introducing new coordinates z_2, z_3, z_4, the blowup of $\mathbb{P}^4$ at p_3 and p_4 can be considered as the change of variables

$$\begin{aligned} y_1 &= 1, & y_2 &= z_2(y_3^2 - 1), \\ y_4 &= z_4(y_3^2 - 1), & y_5 &= -z_3(y_3^2 - 1) + y_3. \end{aligned}$$

Plugging this into (3.1), one can compute

$$v\left(\frac{f}{(y_2 - y_4)^3}\right) = -2, \qquad v\left(\frac{q}{(y_2 - y_4)^2}\right) = -1.$$

It follows from (3.1) that

$$\partial_v(\alpha) = -(z_2 - z_4)^2$$

is trivial in $\kappa(v)^\times/(\kappa(v)^\times)^2$.

Exceptional divisors of $\mathrm{Bl}_{p_5,p_6}(X)$**.** Let v be the valuation on $k(X)^G$ corresponding to the exceptional divisors of $\mathrm{Bl}_{p_5,p_6}(X)$. Similarly as before, after choosing an appropriate affine chart and introducing new coordinates z_1, z_4, z_5, the blowup of $\mathbb{P}^4$ at p_5 and p_6 can be considered as the change of variables

$$y_2 = 1, \qquad y_i = z_i(y_3^2 - 1), \quad i = 1, 4, 5.$$

Plugging this into (3.1), one can compute

$$v\left(\frac{f}{(y_2 - y_4)^3}\right) = v\left(\frac{q}{(y_2 - y_4)^2}\right) = 1.$$

Then $\partial_v(\alpha)$ is obtained by evaluating

$$\frac{f}{q(1 - z_4(y_3^2 - 1))}$$

at $y^3 - 1 = 0$. After the above change of variables, this gives $\partial_v(\alpha) = -1$, and thus we know $\partial_v(\alpha) = 0$.

In summary, condition (3.2) is satisfied for all divisors on $\tilde{X}/G$, hence α defines an element of $\mathrm{Br}([\tilde{X}/G])$. Moreover, it is nonzero since its residue at $w_3 = 0$ is nonzero by (3.4). Since G is a cyclic group, one has

$$\mathrm{H}^2(G, k^\times) = \mathrm{H}^3(G, k^\times) = 0.$$

The sequence (1.3) implies that

$$\mathrm{H}^1(G, \mathrm{Pic}(\tilde{X})) = \mathrm{Br}([\tilde{X}/G]) \tag{3.7}$$

so that the G-action on X is not (stably) linearizable.

Remark 3.2. For the G-action on X given in Proposition 3.1, it is also computed in [6] that

$$\mathrm{H}^1(G, \mathrm{Pic}(\hat{X})) = \mathrm{H}^1(G, \mathrm{Cl}(X)) = \mathbb{Z}/2\mathbb{Z} \tag{3.8}$$

where $\hat{X}$ is the blowup of X at $p_1, \dots, p_6$. In particular, the divisor class group $\mathrm{Cl}(X)$ of X is generated by the class F of a general hyperplane section on X and two classes of rational normal cubic scrolls S_1 and S_2 in X subject to the relation $S_1 + S_2 = 2F$. The G-action switches S_1 and S_2, contributing to nontrivial cohomology (3.8).

Our computation above illustrates how to find nontrivial elements in the group $\mathrm{Br}([\tilde{X}/G])$ via residues in Galois cohomology, without using information of $\mathrm{Pic}(\hat{X})$ as in [6] or the group-theoretic formulas as in [12].

On the other hand, our computation reflects a striking similarity with the computation in [6], making the equality (3.7) explicit. Indeed, with the notation in Proposition 3.1, the factor $q = -y_1y_4 + y_1y_2 - y_2y_4$ in α is a quadric section (equivalent to $2F$ in $\mathrm{Cl}(X)$) cutting out two cubic scrolls on X:

$$X \cap \{q = 0\} = R_1 + R_2,$$

where

$$\begin{aligned} R_1 &= \{q = y_2y_5 + y_1(\sqrt{5}y_2 - y_3 + y_5) = y_3y_4 - y_1(y_3 - \sqrt{5}y_4 - y_5) = 0\},\\ R_2 &= \{q = y_2y_5 - y_1(\sqrt{5}y_2 + y_3 - y_5) = y_3y_4 - y_1(y_3 + \sqrt{5}y_4 - y_5) = 0\}. \end{aligned}$$

One sees that R_1 and R_2 are two rational normal cubic scrolls, corresponding to the two classes S_1 and S_2 in $\mathrm{Cl}(X)$ which contribute to (3.8).

4. Rational surfaces

Throughout this section, we work over $k = \mathbb{C}$. Let $\mathrm{Cr}_2(\mathbb{C})$ be the plane Cremona group, i.e., the group of birational automorphisms over $\mathbb{C}$ of $\mathbb{P}^2$. Finite abelian subgroups of $\mathrm{Cr}_2(\mathbb{C})$ have been classified in [1]. We recall the basic settings. Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a finite group. It is known that we can find a smooth projective surface X with a regular *G-minimal* action on X inducing the embedding $G \subset \mathrm{Cr}_2(\mathbb{C})$. Here a G-minimal action means one of the following two cases holds

(1) $\mathrm{Pic}(X)^G = \mathbb{Z}$ and X is a del Pezzo surface;
(2) $\mathrm{Pic}(X)^G = \mathbb{Z}^2$ and X is a G-conic bundle.

In the remainder of this section, we compute $\mathrm{Br}([X/G])$ in case (1), i.e., for G-minimal del Pezzo surfaces X using Proposition 2.2. We focus on the cases when G is a finite abelian group, and rely on a classification of such models, in particular the lists of groups and *regular* actions in [1, Chapter 10]. We omit technical details of the computation and refer readers to Example 2.3 for an illustration of the computation process. We keep the labels and notation as in *loc. cit.* In particular,

- $L_d(x_1, \dots, x_n)$ denotes a general homogeneous form of degree d in variables $x_1, \dots, x_n$;
- ζ_n is a primitive n-th root of unity;
- λ, μ are general complex numbers.

4.1. **Cyclic groups.** Note that for an action of a cyclic group G on a smooth projective variety X, we have $\mathrm{H}^2(G, \mathbb{C}^\times) = 0$. Then (1.3) implies that

$$\mathrm{Br}([X/G]) = \mathrm{H}^1(G, \mathrm{Pic}(X)).$$

Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a cyclic group generated by an element σ of order n. By [1, Chapter 10.1], we know that up to conjugation in $\mathrm{Cr}_2(\mathbb{C})$, the embedding $G \subset \mathrm{Cr}_2(\mathbb{C})$ is induced by a G-action on X in one of the following cases:

• **Linear automorphisms.**
$\boxed{0.\mathrm{n}}$ $X = \mathbb{P}^2$ and σ acts via weights $(1, 1, \zeta_n)$. One has $\mathrm{Br}([X/G]) = 0$ in these cases.

• **Involutions.** There are three types of involutions (elements of order 2) in $\mathrm{Cr}_2(\mathbb{C})$, up to conjugation. They are:

$\boxed{\mathrm{C}.2}$ *de Jonquières involutions*: X is a conic bundle and the fixed locus is a hyperelliptic curve of genus $g > 0$. The model is in standard form. We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^{2g}$$

(see [4]).
$\boxed{2.\mathrm{G}}$ *Geiser involutions*: The model is given by

$$X = \{w^2 = L_4(x, y, z)\} \subset \mathbb{P}(2, 1, 1, 1)_{w,x,y,z},$$
$$G = \langle\sigma\rangle = C_2, \quad \sigma\colon (w, x, y, z) \mapsto (-w, x, y, z).$$

The fixed curves stratification is given by

i	Curve ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w = 0\}$	$C_2 = \langle\sigma\rangle$	G	3	3	yes

We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^6.$$

$\boxed{1.\mathrm{B}}$ *Bertini involutions*: The model is given by

$$X = \{w^2 = z^3 + L_2(x, y)z^2 + L_4(x, y)z + L_6(x, y)\} \subset \mathbb{P}(3, 1, 1, 2)_{w,x,y,z},$$
$$G = \langle\sigma\rangle = C_2, \quad \sigma\colon (w, x, y, z) \mapsto (-w, x, y, z).$$

The fixed curves stratification is

i	Curve ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w = 0\}$	$C_2 = \langle\sigma\rangle$	G	4	4	yes

We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^8.$$

• **Roots of de Jonquières Involutions.**

C.ro.m and C.re.m X is a conic bundle, σ^m is a de Jonquières involution for some integer m and $n = 2m$. The only stratum with nontrivial stabilizer is a hyperelliptic curve ξ of genus g fixed by σ^m. The model $X \circlearrowleft G$ is in standard form and $\mathrm{Br}([X/G])$ has been computed in [11, Section 5]. The genus of the quotient curve ξ/G depends on the number s of fixed points of σ on ξ. Note that s can be 0, 2 or 4. Let $r = \frac{2g+2}{m}$, we have

$$\mathrm{Br}([X/G]) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{r-2} & \text{if } s = 4, \\ (\mathbb{Z}/2\mathbb{Z})^{r-1} & \text{if } s = 2, \\ (\mathbb{Z}/2\mathbb{Z})^{r} & \text{if } s = 0. \end{cases}$$

• **n=3.**

3.3 The model is given by

$$X = \{w^3 = L_3(x, y, z)\} \subset \mathbb{P}^3_{w,x,y,z},$$

$$G = \langle\sigma\rangle = C_3, \quad \sigma\colon (w, x, y, z) \mapsto (\zeta_3 w, x, y, z).$$

The fixed curves stratification is

i	Curve ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w = 0\}$	$C_3 = \langle\sigma\rangle$	G	1	1	yes

We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/3\mathbb{Z})^2.$$

1.ρ The model is given by

$$X = \{w^2 = z^3 + L_6(x, y, z)\} \subset \mathbb{P}(3, 1, 1, 2)_{w,x,y,z},$$

$$G = \langle\sigma\rangle = C_3, \quad \sigma\colon (w, x, y, z) \mapsto (w, x, y, \zeta_3 z).$$

The fixed curves stratification is

i	Curve ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{z = 0\}$	$C_3 = \langle\sigma\rangle$	G	2	2	yes

We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/3\mathbb{Z})^4.$$

• **n=4.**

2.4 The model is given by

$$X = \{w^2 = L_4(x, y) + z^4\} \subset \mathbb{P}(2, 1, 1, 1)_{w,x,y,z},$$

$$G = \langle\sigma\rangle = C_4, \quad \sigma\colon (w, x, y, z) \mapsto (w, x, y, \zeta_4 z).$$

The fixed curves stratification is

i	Curve ξ_i	I_{ξ_i}	D_{ξ_i}	g(ξ_i)	g(ξ_i/D_{ξ_i})	Standard form
1	$\{z=0\}$	$C_4=\langle\sigma\rangle$	G	1	1	yes

We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/4\mathbb{Z})^2.$$

1.B2.2 The model is given by

$$X = \{w^2 = z^3 + zL_2(x^2, y^2) + xyL_2'(x^2, y^2)\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z},$$

$$G = \langle\sigma\rangle = C_4, \quad \sigma\colon (w,x,y,z) \mapsto (\zeta_4 w, x, -y, -z).$$

The fixed curves stratification is

i	Curve ξ_i	I_{ξ_i}	D_{ξ_i}	g(ξ_i)	g(ξ_i/D_{ξ_i})	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma^2\rangle$	G	4	2	yes

We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^4.$$

• **n=5.**

1.5 The model is given by

$$X = \{w^2 = z^3 + \lambda x^4 z + x(\mu x^5 + y^5)\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z},$$

$$G = \langle\sigma\rangle = C_5, \quad \sigma\colon (w,x,y,z) \mapsto (w, x, \zeta_5 y, z).$$

The fixed curves stratification is

i	Curve ξ_i	I_{ξ_i}	D_{ξ_i}	g(ξ_i)	g(ξ_i/D_{ξ_i})	Standard form
1	$\{y=0\}$	$C_5=\langle\sigma\rangle$	G	1	1	yes

We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/5\mathbb{Z})^2.$$

• **n=6.**

3.6.1 The model is given by

$$X = \{w^3 + x^3 + y^3 + xz^2 + \lambda yz^2 = 0\} \subset \mathbb{P}^3_{w,x,y,z},$$

$$G = \langle\sigma\rangle = C_6, \quad \sigma\colon (w,x,y,z) \mapsto (\zeta_3 w, x, y, -z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	g(ξ_i)	g(ξ_i/D_{ξ_i})	Standard form
1	$\{z=0\}$	$C_2=\langle\sigma^3\rangle$	G	1	0	yes
2	$\{w=0\}$	$C_3=\langle\sigma^2\rangle$	G	1	0	

We have

$$\mathrm{Br}([X/G]) = 0.$$

3.6.2 The model is given by

$$X = \{wx^2 + w^3 + y^3 + z^3 + \lambda wyz = 0\} \subset \mathbb{P}^3_{w,x,y,z},$$

$$G = \langle \sigma \rangle = C_6, \quad \sigma \colon (w,x,y,z) \mapsto (w, -x, \zeta_3 y, \zeta_3^2 z).$$

The fixed curves stratification is

i	Curve ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{x=0\}$	$C_2 = \langle \sigma^3 \rangle$	G	1	1	yes

We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^2.$$

2.G3.1 The model is given by

$$X = \{w^2 = L_4(x,y) + z^3 L_1(x,y)\} \subset \mathbb{P}(2,1,1,1)_{w,x,y,z},$$

$$G = \langle \sigma \rangle = C_6, \quad \sigma \colon (w,x,y,z) \mapsto (-w, x, y, \zeta_3 z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2 = \langle \sigma^3 \rangle$	G	3	0	yes
2	$\{z=0\}$	$C_3 = \langle \sigma^2 \rangle$	G	1	0	

We have

$$\mathrm{Br}([X/G]) = 0.$$

2.G3.2 The model is given by

$$X = \{w^2 = x(x^3 + y^3 + z^3) + yzL_1(x^2, yz)\} \subset \mathbb{P}(2,1,1,1)_{w,x,y,z},$$

$$G = \langle \sigma \rangle = C_6, \quad \sigma \colon (w,x,y,z) \mapsto (-w, x, \zeta_3 y, \zeta_3^2 z).$$

The fixed curves stratification is

i	Curve ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2 = \langle \sigma^3 \rangle$	G	3	1	yes

We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^2.$$

2.6 The model is given by

$$X = \{w^2 = x^3 y + y^4 + z^4 + \lambda y^2 z^2\} \subset \mathbb{P}(2,1,1,1)_{w,x,y,z},$$

$$G = \langle \sigma \rangle = C_6, \quad \sigma \colon (w,x,y,z) \mapsto (-w, \zeta_3 x, y, -z).$$

The fixed curves stratification is

i	Curve ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{x=0\}$	$C_3=\langle\sigma^2\rangle$	G	1	1	yes

We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/3\mathbb{Z})^2.$$

1.$\sigma\rho$ The model is given by

$$X = \{w^2 = z^3 + L_6(x,y)\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z},$$
$$G = \langle\sigma\rangle = C_6, \quad \sigma\colon (w,x,y,z) \mapsto (-w,x,y,\zeta_3 z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma^3\rangle$	G	4	0	yes
2	$\{z=0\}$	$C_3=\langle\sigma^2\rangle$	G	2	0	

We have

$$\mathrm{Br}([X/G]) = 0.$$

1.ρ2 The model is given by

$$X = \{w^2 = z^3 + L_3(x^2,y^2)\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z},$$
$$G = \langle\sigma\rangle = C_6, \quad \sigma\colon (w,x,y,z) \mapsto (w,x,-y,\zeta_3 z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{y=0\}$	$C_2=\langle\sigma^3\rangle$	G	1	0	yes
2	$\{z=0\}$	$C_3=\langle\sigma^2\rangle$	G	2	0	

We have

$$\mathrm{Br}([X/G]) = 0.$$

1.B3.1 The model is given by

$$X = \{w^2 = z^3 + xL_1(x^3,y^3)z + L_2(x^3,y^3)\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z},$$
$$G = \langle\sigma\rangle = C_6, \quad \sigma\colon (w,x,y,z) \mapsto (-w,x,\zeta_3 y,z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma^3\rangle$	G	4	0	yes
2	$\{y=0\}$	$C_3=\langle\sigma^2\rangle$	G	1	0	

We have

$$\mathrm{Br}([X/G]) = 0.$$

1.B3.2 The model is given by

$$X = \{w^2 = z^3 + \lambda x^2y^2z + L_2(x^3, y^3)\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z},$$
$$G = \langle\sigma\rangle = C_6, \quad \sigma\colon (w,x,y,z) \mapsto (-w, x, \zeta_3 y, \zeta_3 z).$$

The fixed curves stratification is

i	Curve ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2 = \langle\sigma^3\rangle$	G	4	2	yes

We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^4.$$

1.6 The model is given by

$$X = \{w^2 = z^3 + \lambda x^4 z + \mu x^6 + y^6\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z},$$
$$G = \langle\sigma\rangle = C_6, \quad \sigma\colon (w,x,y,z) \mapsto (w, x, -\zeta_3 y, z).$$

The fixed curves stratification is

i	Curve ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_6 = \langle\sigma\rangle$	G	1	1	yes

We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/6\mathbb{Z})^2.$$

• **n=8.**

1.B4.2 The model is given by

$$X = \{w^2 = \lambda x^2y^2z + xy(x^4+y^4)\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z},$$
$$G = \langle\sigma\rangle = C_8, \quad \sigma\colon (w,x,y,z) \mapsto (\zeta_8 w, x, \zeta_4 y, -\zeta_4 z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	Standard form
1	$\{x=0\}$	$C_2 = \langle\sigma^4\rangle$	G	0	no
2	$\{y=0\}$	$C_2 = \langle\sigma^4\rangle$	G	0	
3	$\{\lambda xyz + x^4 + y^4 = 0\}$	$C_2 = \langle\sigma^4\rangle$	G	0	

The model is not in standard form: the divisor $\{w=0\}\cap X$ fixed by σ^4 is the union of three rational curves ξ_1, ξ_2 and ξ_3 meeting at one point $p = [0:0:0:1]$, and thus not normal crossing. Moreover, p is a node of ξ_3. To reach a standard form, consider the blowup of X at p. Let E_1 be the exceptional divisor and $\tilde{\xi}_i$ be the strict transform of ξ_i for $i = 1, 2, 3$. We find $\tilde{\xi}_3$ meets E_1 at two points p_1 and p_2, $\tilde{\xi}_1 \cap \tilde{\xi}_3 \cap E_1 = \{p_1\}$, $\tilde{\xi}_2 \cap \tilde{\xi}_3 \cap E_1 = \{p_2\}$ and $\tilde{\xi}_1$ and $\tilde{\xi}_2$ are disjoint. Then blowing up the points p_1 and p_2 brings the model into a standard form. One then computes via Proposition 2.2 that

$$\mathrm{Br}([X/G]) = \mathbb{Z}/2\mathbb{Z}.$$

• **n=9.**

$\boxed{3.9}$ The model is given by

$$X = \{w^3 + xz^2 + x^2y + y^2z = 0\} \subset \mathbb{P}^3_{w,x,y,z},$$

$$G = \langle\sigma\rangle = C_9, \quad \sigma\colon (w, x, y, z) \mapsto (\zeta_9 w, x, \zeta_3 y, \zeta_3^2 z).$$

The fixed curves stratification is given by

i	Curve ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w = 0\}$	$C_3 = \langle\sigma^3\rangle$	G	1	0	yes

We have

$$\mathrm{Br}([X/G]) = 0.$$

• **n=10.**

$\boxed{1.B5}$ The model is given by

$$X = \{w^2 = z^3 + \lambda x^4 z + x(\mu x^5 + y^5)\} \subset \mathbb{P}(3, 1, 1, 2)_{w,x,y,z},$$

$$G = \langle\sigma\rangle = C_{10}, \quad \sigma\colon (w, x, y, z) \mapsto (-w, x, \zeta_5 y, z).$$

The fixed curves stratification is given by

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w = 0\}$	$C_2 = \langle\sigma^5\rangle$	G	4	0	yes
2	$\{y = 0\}$	$C_5 = \langle\sigma^2\rangle$	G	1	0	

We have

$$\mathrm{Br}([X/G]) = 0.$$

• **n=12.**

$\boxed{3.12}$ The model is given by

$$X = \{w^3 + x^3 + yz^2 + y^2x = 0\} \subset \mathbb{P}^3_{w,x,y,z},$$

$$G = \langle\sigma\rangle = C_{12}, \quad \sigma\colon (w, x, y, z) \mapsto (\zeta_3 w, x, -y, \zeta_4 z).$$

The fixed curves stratification is given by

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$g(\xi_i)$	$g(\xi_i/D_{\xi_i})$	Standard form
1	$\{z=0\}$	$C_2=\langle\sigma^6\rangle$	G	1	0	yes
2	$\{w=0\}$	$C_3=\langle\sigma^4\rangle$	G	1	0	

We have

$$\mathrm{Br}([X/G])=0.$$

2.12 The model is given by

$$X=\{w^2=x^3y+y^4+z^4\}\subset\mathbb{P}(2,1,1,1)_{w,x,y,z},$$

$$G=\langle\sigma\rangle=C_{12},\quad \sigma\colon(w,x,y,z)\mapsto(w,\zeta_3x,y,\zeta_4z).$$

The fixed curves stratification is given by

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$g(\xi_i)$	$g(\xi_i/D_{\xi_i})$	Standard form
1	$\{z=0\}$	$C_2=\langle\sigma^6\rangle$	G	1	0	yes
2	$\{x=0\}$	$C_3=\langle\sigma^4\rangle$	G	1	0	

We have

$$\mathrm{Br}([X/G])=0.$$

$1.\sigma\rho2.2$ The model is given by

$$X=\{w^2=z^3+xyL_2(x^2,y^2)\}\subset\mathbb{P}(3,1,1,2)_{w,x,y,z},$$

$$G=\langle\sigma\rangle=C_{12},\quad \sigma\colon(w,x,y,z)\mapsto(\zeta_4w,x,-y,-\zeta_3z).$$

The fixed curves stratification is given by

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$g(\xi_i)$	$g(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma^6\rangle$	G	4	0	yes
2	$\{z=0\}$	$C_3=\langle\sigma^4\rangle$	G	2	0	

We have

$$\mathrm{Br}([X/G])=0.$$

- **n=14.**

2.G7 The model is given by

$$X=\{w^2=x^3y+y^3z+xz^3\}\subset\mathbb{P}(2,1,1,1)_{w,x,y,z},$$

$$G=\langle\sigma\rangle=C_{14},\quad \sigma\colon(w,x,y,z)\mapsto(-w,\zeta_7x,\zeta_7^4y,\zeta_7^2z).$$

The fixed curves stratification is given by

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$g(\xi_i)$	$g(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma^7\rangle$	G	3	0	yes

We have

$$\mathrm{Br}([X/G])=0.$$

• n=15.

1.ρ5 The model is given by

$$X=\{w^2=z^3+x(x^5+y^5)\}\subset\mathbb{P}(3,1,1,2)_{w,x,y,z},$$
$$G=\langle\sigma\rangle=C_{15},\quad \sigma\colon(w,x,y,z)\mapsto(w,x,\zeta_5 y,\zeta_3 z).$$

The fixed curves stratification is given by

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$g(\xi_i)$	$g(\xi_i/D_{\xi_i})$	Standard form
1	$\{z=0\}$	$C_3=\langle\sigma^5\rangle$	G	2	0	yes
2	$\{y=0\}$	$C_5=\langle\sigma^3\rangle$	G	1	0	

We have

$$\mathrm{Br}([X/G])=0.$$

• n=18.

2.G9 The model is given by

$$X=\{w^2=x^3y+y^4+xz^3\}\subset\mathbb{P}(2,1,1,1)_{w,x,y,z},$$
$$G=\langle\sigma\rangle=C_{18},\quad \sigma\colon(w,x,y,z)\mapsto(-w,\zeta_9^6 x,y,\zeta_9 z).$$

The fixed curves stratification is given by

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$g(\xi_i)$	$g(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma^9\rangle$	G	3	0	yes
2	$\{z=0\}$	$C_3=\langle\sigma^6\rangle$	G	1	0	

We have

$$\mathrm{Br}([X/G])=0.$$

• n=20.

1.B10 The model is given by

$$X=\{w^2=z^3+x^4z+xy^5\}\subset\mathbb{P}(3,1,1,2)_{w,x,y,z},$$
$$G=\langle\sigma\rangle=C_{20},\quad \sigma\colon(w,x,y,z)\mapsto(\zeta_4 w,x,\zeta_{10}y,-z).$$

The fixed curves stratification is given by

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{z=0\}$	$C_2=\langle\sigma^{10}\rangle$	G	0	0	no
2	$\{y=0\}$	$C_5=\langle\sigma^4\rangle$	G	1	0	

The model is not in standard form. The curve ξ_1 has an A_4-singularity at $p=[0:1:0:0]$, and ξ_1 intersects ξ_2 at p non-transversally. One can obtain a standard form via successive blowups such that the strict transforms of ξ_1 and ξ_2 and the exceptional divisors form a tree of rational curves. We have

$$\mathrm{Br}([X/G])=0.$$

• n=24.

$\boxed{1.\sigma\rho 4}$ The model is given by

$$X=\{w^2=z^3+xy(x^4+y^4)\}\subset\mathbb{P}(3,1,1,2)_{w,x,y,z},$$

$$G=\langle\sigma\rangle=C_{24},\quad \sigma\colon(w,x,y,z)\mapsto(\zeta_8 w,x,\zeta_4 y,-\zeta_{12}^7 z).$$

The fixed curves stratification is given by

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma^{12}\rangle$	G	4	0	yes
2	$\{z=0\}$	$C_3=\langle\sigma^8\rangle$	G	2	0	

We have

$$\mathrm{Br}([X/G])=0.$$

• n=30.

$\boxed{1.\sigma\rho 5}$ The model is given by

$$X=\{w^2=z^3+x(x^5+y^5)\}\subset\mathbb{P}(3,1,1,2)_{w,x,y,z},$$

$$G=\langle\sigma\rangle=C_{30},\quad \sigma\colon(w,x,y,z)\mapsto(-w,x,\zeta_5 y,\zeta_3 z).$$

The fixed curves stratification is given by

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma^{15}\rangle$	G	4	0	yes
2	$\{z=0\}$	$C_3=\langle\sigma^{10}\rangle$	G	2	0	
3	$\{y=0\}$	$C_5=\langle\sigma^6\rangle$	G	1	0	

We have

$$\mathrm{Br}([X/G])=0.$$

4.2. **Noncyclic groups.** We continue with actions of noncyclic groups.

• **Automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$**
Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. Note that $\mathrm{H}^1(G, \mathrm{Pic}(X)) = 0$ for any $G \subset \mathrm{Aut}(X)$ and thus

$$\mathrm{Br}([X/G]) = \mathrm{H}^2(G, \mathbb{C}^\times)/\mathrm{Am}(X, G),$$

where $\mathrm{Am}(X, G)$ is the Amitsur group of the G-action on X. The computation of $\mathrm{Am}(X, G)$ in this case is straightforward, see e.g., [3, Proposition 6.7]. So for actions on quadric surfaces, we compute $\mathrm{Br}([X/G])$ from the Amitsur groups.

0.mn The action on $X = \mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$G = C_n \times C_m, \qquad (x, y) \overset{\sigma_1}{\mapsto} (\zeta_n x, y), \quad (x, y) \overset{\sigma_2}{\mapsto} (x, \zeta_m y).$$

One has $\mathrm{Pic}(X)^G = \mathbb{Z}^2$, generated by the G-invariant line bundles $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$. Both $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$ are G-linearizable. It follows that $\mathrm{Am}(X, G) = 0$ and

$$\mathrm{Br}([X/G]) = \mathrm{H}^2(G, \mathbb{C}^\times) = \mathbb{Z}/\mathrm{gcd}(n, m)\mathbb{Z}.$$

We also compute the fixed curves stratification in this case

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$[1:0] \times \mathbb{P}^1$	$C_n = \langle \sigma_1 \rangle$	G	0	0	yes
2	$[0:1] \times \mathbb{P}^1$	$C_n = \langle \sigma_1 \rangle$	G	0	0	
3	$\mathbb{P}^1 \times [1:0]$	$C_m = \langle \sigma_2 \rangle$	G	0	0	
4	$\mathbb{P}^1 \times [0:1]$	$C_m = \langle \sigma_2 \rangle$	G	0	0	

Using Proposition 2.2, one can also deduce $\mathrm{Br}([X/G]) = \mathbb{Z}/\mathrm{gcd}(n, m)\mathbb{Z}$.

P1.22n The action on $X = \mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$G = C_2 \times C_{2n}, \qquad (x, y) \mapsto (x^{-1}, y), \quad (x, y) \mapsto (-x, \zeta_{2n} y).$$

One has $\mathrm{Pic}(X)^G = \mathbb{Z}^2$, generated by $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$. The line bundle $\mathcal{O}(1, 0)$ is not G-linearizable while $\mathcal{O}(0, 1)$ is. It follows that $\mathrm{Am}(X, G) = \mathbb{Z}/2\mathbb{Z}$ and

$$\mathrm{Br}([X/G]) = 0.$$

P1.222n The action on $X = \mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$G = C_2^2 \times C_{2n}, \qquad (x, y) \mapsto (\pm x^{\pm 1}, y), \quad (x, y) \mapsto (x, \zeta_{2n} y).$$

One has $\mathrm{Pic}(X)^G = \mathbb{Z}^2$, generated by $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$. The line bundle $\mathcal{O}(1,0)$ is not G-linearizable while $\mathcal{O}(0,1)$ is. It follows that $\mathrm{Am}(X,G) = \mathbb{Z}/2\mathbb{Z}$ and

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^2.$$

P1.22.1 The action on $X = \mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$G = C_2^2, \qquad (x,y) \mapsto (\pm x^{\pm 1}, y).$$

One has $\mathrm{Pic}(X)^G = \mathbb{Z}^2$, generated by $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$. The line bundle $\mathcal{O}(1,0)$ is not G-linearizable while $\mathcal{O}(0,1)$ is. It follows that $\mathrm{Am}(X,G) = \mathbb{Z}/2\mathbb{Z}$ and

$$\mathrm{Br}([X/G]) = 0.$$

P1.222 The action on $X = \mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$G = C_2^3, \qquad (x,y) \mapsto (\pm x, \pm y), \qquad (x,y) \mapsto (x^{-1}, y).$$

One has $\mathrm{Pic}(X)^G = \mathbb{Z}^2$, generated by $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$. The line bundle $\mathcal{O}(1,0)$ is not G-linearizable while $\mathcal{O}(0,1)$ is. It follows that $\mathrm{Am}(X,G) = \mathbb{Z}/2\mathbb{Z}$ and

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^2.$$

P1.2222 The action on $X = \mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$G = C_2^4, \qquad (x,y) \mapsto (\pm x^{\pm 1}, \pm y^{\pm 1}).$$

One has $\mathrm{Pic}(X)^G = \mathbb{Z}^2$, generated by $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$. Both $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$ are not G-linearizable. It follows that $\mathrm{Am}(X,G) = (\mathbb{Z}/2\mathbb{Z})^2$ and

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^4.$$

P1s.24 The action on $X = \mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$G = C_2 \times C_4, \qquad (x,y) \mapsto (x^{-1}, y^{-1}), \qquad (x,y) \mapsto (-y, x).$$

One has $\mathrm{Pic}(X)^G = \mathbb{Z}$, generated by $\mathcal{O}(1,1)$, which is not G-linearizable. It follows that $\mathrm{Am}(X,G) = \mathbb{Z}/2\mathbb{Z}$ and

$$\mathrm{Br}([X/G]) = 0.$$

P1s.222 The action on $X = \mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$G = C_2^3, \qquad (x,y) \mapsto (-x,-y), \quad (x,y) \mapsto (x^{-1}, y^{-1}), \quad (x,y) \mapsto (y,x).$$

One has $\mathrm{Pic}(X)^G = \mathbb{Z}$, generated by $\mathcal{O}(1,1)$, which is G-linearizable. It follows that $\mathrm{Am}(X,G) = 0$ and

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^3.$$

• **Automorphisms of** $\mathbb{P}^2$

0.V9 The action on $X = \mathbb{P}^2$ is given by

$$G = C_3^2, \qquad (x:y:z) \mapsto (x : \zeta_3 y : \zeta_3^2 z), \quad (x:y:z) \mapsto (y:z:x).$$

One has $\mathrm{Pic}(X)^G = \mathbb{Z}$, generated by $\mathcal{O}(1)$, which is not G-linearizable. It follows that $\mathrm{Am}(X,G) = \mathbb{Z}/3\mathbb{Z}$ and

$$\mathrm{Br}([X/G]) = 0.$$

• **Automorphisms of del Pezzo surfaces of degree** 4

The surface $X \subset \mathbb{P}^4_{x_1,\ldots,x_5}$ is given by the following equations with general $a, b, c \in \mathbb{C}$

$$\begin{aligned} cx_1^2 - ax_3^2 - (a-c)x_4^2 - ac(a-c)x_5^2 &= 0 \\ cx_2^2 - bx_3^2 + (c-b)x_4^2 - bc(c-b)x_5^2 &= 0. \end{aligned} \tag{4.1}$$

4.222 The action on X is given by

$$G = C_2^3, \qquad \sigma_1 : (\mathbf{x}) \mapsto (-x_1, x_2, x_3, x_4, x_5),$$

$$\sigma_2 : (\mathbf{x}) \mapsto (x_1, -x_2, x_3, x_4, x_5), \quad \sigma_3 : (\mathbf{x}) \mapsto (x_1, x_2, -x_3, x_4, x_5).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{x_1 = 0\}$	$C_2 = \langle\sigma_1\rangle$	G	1	0	
2	$\{x_2 = 0\}$	$C_2 = \langle\sigma_2\rangle$	G	1	0	yes
3	$\{x_3 = 0\}$	$C_2 = \langle\sigma_3\rangle$	G	1	0	

The images of ξ_i and ξ_j in $X^{(1)}/G$ intersect in 2 points for $i \neq j \in \{1,2,3\}$. We find

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^4.$$

4.2222 The action on X is given by

$$G = C_2^4, \quad \sigma_1 : (\mathbf{x}) \mapsto (-x_1, x_2, x_3, x_4, x_5), \quad \sigma_2 : (\mathbf{x}) \mapsto (x_1, -x_2, x_3, x_4, x_5),$$

$$\sigma_3 : (\mathbf{x}) \mapsto (x_1, x_2, -x_3, x_4, x_5), \quad \sigma_4 : (\mathbf{x}) \mapsto (x_1, x_2, x_3, -x_4, x_5).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$g(\xi_i)$	$g(\xi_i/D_{\xi_i})$	Standard form
1	$\{x_1 = 0\}$	$C_2 = \langle \sigma_1 \rangle$	G	1	0	
2	$\{x_2 = 0\}$	$C_2 = \langle \sigma_2 \rangle$	G	1	0	
3	$\{x_3 = 0\}$	$C_2 = \langle \sigma_3 \rangle$	G	1	0	yes
4	$\{x_4 = 0\}$	$C_2 = \langle \sigma_4 \rangle$	G	1	0	
5	$\{x_5 = 0\}$	$C_2 = \langle \sigma_1\sigma_2\sigma_3\sigma_4 \rangle$	G	1	0	

The images of ξ_i and ξ_j in $X^{(1)}/G$ intersect in 1 point for $i \neq j \in \{1, \ldots, 5\}$. We find

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^6.$$

4.42 The surface X is given by (4.1) with $(a : b : c) = (1 : \xi : 1 + \xi)$ for any $\xi \in \mathbb{C} \setminus \{0, \pm 1\}$. The $G = C_4 \times C_2$-action on X is generated by

$$\sigma_1 : (\mathbf{x}) \mapsto (-x_2, x_1, x_4, x_3, -x_5), \qquad \sigma_2 : (\mathbf{x}) \mapsto (x_1, x_2, x_3, x_4, -x_5).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$g(\xi_i)$	$g(\xi_i/D_{\xi_i})$	Standard form
1	$\{x_5 = 0\}$	$C_2 = \langle \sigma_2 \rangle$	G	1	1	yes

We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2)^2.$$

- **Automorphisms of cubic surfaces**

3.33.1 The model is given by

$$X = \{w^3 + x^3 + y^3 + z^3 = 0\} \subset \mathbb{P}^3_{w,x,y,z}, \quad G = \langle \sigma_1, \sigma_2 \rangle = C_3^2,$$

$$\sigma_1 : (w, x, y, z) \mapsto (\zeta_3 w, x, y, z), \quad \sigma_2 : (w, x, y, z) \mapsto (w, x, y, \zeta_3 z).$$

The group

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/3\mathbb{Z})^2$$

has been computed in [11, Section 5].

3.33.2 The model is given by

$$X = \{w^3 + x^3 + y^3 + z^3 + \lambda xyz = 0\} \subset \mathbb{P}^3_{w,x,y,z}, \quad G = \langle \sigma_1, \sigma_2 \rangle = C_3^2,$$

$$\sigma_1 : (w, x, y, z) \mapsto (\zeta_3 w, x, y, z), \quad \sigma_2 : (w, x, y, z) \mapsto (w, x, \zeta_3 y, \zeta_3^2 z).$$

The fixed curves stratification is given by

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$g(\xi_i)$	$g(\xi_i/D_{\xi_i})$	Standard form
1	$\{w = 0\}$	$C_3 = \langle \sigma_1 \rangle$	G	1	1	yes

and

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/3\mathbb{Z})^2.$$

3.36 The model is given by

$$X = \{w^3 + x^3 + xy^2 + z^3 = 0\} \subset \mathbb{P}^3_{w,x,y,z}, \quad G = \langle \sigma_1, \sigma_2 \rangle = C_3 \times C_6,$$

$$\sigma_1 : (w,x,y,z) \mapsto (\zeta_3 w, x, y, z), \quad \sigma_2 : (w,x,y,z) \mapsto (w, x, -y, \zeta_3 z).$$

The fixed curves stratification is given by

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{y=0\}$	$C_2 = \langle \sigma_2^3 \rangle$	G	1	0	
2	$\{w=0\}$	$C_3 = \langle \sigma_1 \rangle$	G	1	0	yes
3	$\{z=0\}$	$C_3 = \langle \sigma_2^2 \rangle$	G	1	0	

The images of ξ_2 and ξ_3 in $X^{(1)}/G$ intersect in two points, so that

$$\mathrm{Br}([X/G]) = \mathbb{Z}/3\mathbb{Z}.$$

3.333 The model is given by

$$X = \{w^3 + x^3 + y^3 + z^3 = 0\} \subset \mathbb{P}^3_{w,x,y,z},$$

$$G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle = C_3^3, \quad \sigma_1 : (w,x,y,z) \mapsto (\zeta_3 w, x, y, z),$$

$$\sigma_2 : (w,x,y,z) \mapsto (w, x, \zeta_3 y, z), \quad \sigma_3 : (w,x,y,z) \mapsto (w, x, y, \zeta_3 z).$$

The group

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/3\mathbb{Z})^3$$

has been computed in [11, Section 5].

• Automorphisms of Del Pezzo surfaces of degree 2

2.G2 The model is given by

$$X = \{w^2 = L_4(x,y) + L_2(x,y)z^2 + z^4\} \subset \mathbb{P}(2,1,1,1)_{w,x,y,z},$$

$$G = \langle \sigma_1, \sigma_2 \rangle = C_2^2,$$

$$\sigma_1 \colon (w,x,y,z) \mapsto (-w, x, y, z), \quad \sigma_2 \colon (w,x,y,z) \mapsto (w, x, y, -z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2 = \langle \sigma_1 \rangle$	G	3	1	yes
2	$\{z=0\}$	$C_2 = \langle \sigma_2 \rangle$	G	1	0	

The images of ξ_1 and ξ_2 in $X^{(1)}/G$ meet at four points. Recall that a zero-cycle $\sum_i n_i P_i$ of degree 0 on an elliptic curve is a divisor of a function on the curve if and only if $\sum n_i[P_i] = 0$, where the latter sum is for the group law of the elliptic curve. It follows that we have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^5.$$

2.G4.1 The model is given by

$$X = \{w^2 = L_4(x,y) + z^4\} \subset \mathbb{P}(2,1,1,1)_{w,x,y,z}, \quad G = \langle \sigma_1, \sigma_2 \rangle = C_2 \times C_4,$$

$$\sigma_1\colon (w,x,y,z) \mapsto (-w,x,y,z), \quad \sigma_2\colon (w,x,y,z) \mapsto (w,x,y,\zeta_4 z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2 = \langle \sigma_1 \rangle$	G	3	0	yes
2	$\{z=0\}$	$C_4 = \langle \sigma_2 \rangle$	G	1	0	

The images of ξ_1 and ξ_2 in $X^{(1)}/G$ meet at four points. We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^3.$$

2.G4.2 The model is given by

$$X = \{w^2 = x^4 + y^4 + z^4 + xyL_1(xy, z^2)\} \subset \mathbb{P}(2,1,1,1)_{w,x,y,z},$$

$$G = \langle \sigma_1, \sigma_2 \rangle = C_2 \times C_4,$$

$$\sigma_1\colon (w,x,y,z) \mapsto (-w,x,y,z), \quad \sigma_2\colon (w,x,y,z) \mapsto (w,x,-y,\zeta_4 z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2 = \langle \sigma_1 \rangle$	G	3	1	yes
2	$\{z=0\}$	$C_2 = \langle \sigma_2^2 \rangle$	G	1	0	

The images of ξ_1 and ξ_2 in $X^{(1)}/G$ meet at two points. We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^3.$$

2.G6 The model is given by

$$X = \{w^2 = x^3y + y^4 + z^4 + \lambda y^2 z^2\} \subset \mathbb{P}(2,1,1,1)_{w,x,y,z},$$

$$G = \langle \sigma_1, \sigma_2 \rangle = C_2 \times C_6,$$

$$\sigma_1\colon (w,x,y,z) \mapsto (-w,x,y,z), \quad \sigma_2\colon (w,x,y,z) \mapsto (w,\zeta_3 x,y,-z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma_1\rangle$	G	3	0	yes
2	$\{z=0\}$	$C_2=\langle\sigma_2^3\rangle$	G	1	0	
3	$\{x=0\}$	$C_3=\langle\sigma_2^2\rangle$	G	1	0	

The images of ξ_1 and ξ_2 in $X^{(1)}/G$ meet at two points. We have

$$\mathrm{Br}([X/G])=\mathbb{Z}/2\mathbb{Z}.$$

2.G8 The model is given by

$$X=\{w^2=x^3y+xy^3+z^4\}\subset\mathbb{P}(2,1,1,1)_{w,x,y,z},\quad G=\langle\sigma_1,\sigma_2\rangle=C_2\times C_8,$$

$$\sigma_1\colon(w,x,y,z)\mapsto(-w,x,y,z),\quad \sigma_2\colon(w,x,y,z)\mapsto(w,\zeta_8x,-\zeta_8y,z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma_1\rangle$	G	3	0	yes
2	$\{z=0\}$	$C_4=\langle\sigma_1\sigma_2^2\rangle$	G	1	0	

The images of ξ_1 and ξ_2 in $X^{(1)}/G$ meet at three points. We have

$$\mathrm{Br}([X/G])=(\mathbb{Z}/2\mathbb{Z})^2.$$

2.G12 The model is given by

$$X=\{w^2=x^3y+y^4+z^4\}\subset\mathbb{P}(2,1,1,1)_{w,x,y,z},\quad G=\langle\sigma_1,\sigma_2\rangle=C_2\times C_{12},$$

$$\sigma_1\colon(w,x,y,z)\mapsto(-w,x,y,z),\quad \sigma_2\colon(w,x,y,z)\mapsto(w,\zeta_3x,y,\zeta_4z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma_1\rangle$	G	3	0	yes
2	$\{x=0\}$	$C_3=\langle\sigma_2^4\rangle$	G	1	0	
3	$\{z=0\}$	$C_4=\langle\sigma_2^3\rangle$	G	1	0	

The images of ξ_1 and ξ_3 in $X^{(1)}/G$ meet at two points. We have

$$\mathrm{Br}([X/G])=\mathbb{Z}/2\mathbb{Z}.$$

2.G22 The model is given by

$$X=\{w^2=L_2(x^2,y^2,z^2)\}\subset\mathbb{P}(2,1,1,1)_{w,x,y,z},\quad G=\langle\sigma_1,\sigma_2,\sigma_3\rangle=C_2^3,$$

$$\sigma_1\colon(w,x,y,z)\mapsto(-w,x,y,z),\quad \sigma_2\colon(w,x,y,z)\mapsto(w,x,-y,z),$$

$$\sigma_3\colon(w,x,y,z)\mapsto(w,x,y,-z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma_1\rangle$	G	3	0	yes
2	$\{x=0\}$	$C_2=\langle\sigma_2\sigma_3\rangle$	G	1	0	
3	$\{y=0\}$	$C_2=\langle\sigma_2\rangle$	G	1	0	
4	$\{z=0\}$	$C_2=\langle\sigma_3\rangle$	G	1	0	

Let p_{ij} be the number of intersection points of the images of ξ_i and ξ_j in $X^{(1)}/G$. We record

$$p_{12}=p_{13}=p_{14}=2,\quad p_{23}=p_{24}=p_{34}=1.$$

Since the intersection of any three of the four curves ξ_i, $i=1,2,3,4$ is empty, we have

$$\mathrm{Br}([X/G])=(\mathbb{Z}/2\mathbb{Z})^6.$$

2.G24 The model is given by

$$X=\{w^2=x^4+y^4+z^4+\lambda x^2y^2\}\subset\mathbb{P}(2,1,1,1)_{w,x,y,z},$$

$$G=\langle\sigma_1,\sigma_2,\sigma_3\rangle=C_2^2\times C_4,\quad \sigma_1\colon(w,x,y,z)\mapsto(-w,x,y,z),$$

$$\sigma_2\colon(w,x,y,z)\mapsto(w,x,-y,z),\quad \sigma_3\colon(w,x,y,z)\mapsto(w,x,y,\zeta_4 z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma_1\rangle$	G	3	0	yes
2	$\{x=0\}$	$C_2=\langle\sigma_2\sigma_3^2\rangle$	G	1	0	
3	$\{y=0\}$	$C_2=\langle\sigma_2\rangle$	G	1	0	
4	$\{z=0\}$	$C_4=\langle\sigma_3\rangle$	G	1	0	

Let p_{ij} be the number of intersection points of the images of ξ_i and ξ_j in $X^{(1)}/G$. We record

$$p_{14}=2,\quad p_{12}=p_{13}=p_{23}=p_{24}=p_{34}=1.$$

Since the intersection of any three of the four curves ξ_i, $i=1,2,3,4$ is empty, we have

$$\mathrm{Br}([X/G])=(\mathbb{Z}/2\mathbb{Z})^4.$$

2.G44 The model is given by

$$X=\{w^2=x^4+y^4+z^4\}\subset\mathbb{P}(2,1,1,1)_{w,x,y,z},$$

$$G=\langle\sigma_1,\sigma_2,\sigma_3\rangle=C_2\times C_4^2,\quad \sigma_1\colon(w,x,y,z)\mapsto(-w,x,y,z),$$

$$\sigma_2\colon(w,x,y,z)\mapsto(w,x,\zeta_4 y,z),\quad \sigma_3\colon(w,x,y,z)\mapsto(w,x,y,\zeta_4 z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma_1\rangle$	G	3	0	yes
2	$\{x=0\}$	$C_4=\langle\sigma_1\sigma_2\sigma_3\rangle$	G	1	0	
3	$\{y=0\}$	$C_4=\langle\sigma_2\rangle$	G	1	0	
4	$\{z=0\}$	$C_4=\langle\sigma_3\rangle$	G	1	0	

Let p_{ij} be the number of intersection points of the images of ξ_i and ξ_j in $X^{(1)}/G$. We record

$$p_{12}=p_{13}=p_{14}=p_{23}=p_{24}=p_{34}=1.$$

Since the intersection of any three of the four curves ξ_i, $i=1,2,3,4$ is empty, we have

$$\mathrm{Br}([X/G])=(\mathbb{Z}/2\mathbb{Z})^2\oplus(\mathbb{Z}/4\mathbb{Z}).$$

2.24.1 The model is given by

$$X=\{w^2=x^4+y^4+z^4+\lambda x^2y^2\}\subset\mathbb{P}(2,1,1,1)_{w,x,y,z},$$

$$G=\langle\sigma_1,\sigma_2\rangle=C_2\times C_4,\quad \sigma_1\colon(w,x,y,z)\mapsto(w,x,-y,z),$$

$$\sigma_2\colon(w,x,y,z)\mapsto(w,x,y,\zeta_4z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{y=0\}$	$C_2=\langle\sigma_1\rangle$	G	1	0	yes
2	$\{x=0\}$	$C_2=\langle\sigma_1\sigma_2^2\rangle$	G	1	0	
3	$\{z=0\}$	$C_4=\langle\sigma_2\rangle$	G	1	0	

Let p_{ij} be the number of intersection points of the images of ξ_i and ξ_j in $X^{(1)}/G$. We record

$$p_{13}=p_{23}=2,\quad p_{12}=1.$$

Since $\xi_1\cap\xi_2\cap\xi_3$ is empty, we have

$$\mathrm{Br}([X/G])=(\mathbb{Z}/2\mathbb{Z})^3.$$

2.24.2 The model is given by

$$X=\{w^2=x^4+y^4+z^4+\lambda x^2y^2\}\subset\mathbb{P}(2,1,1,1)_{w,x,y,z},$$

$$G=\langle\sigma_1,\sigma_2\rangle=C_2\times C_4,\quad \sigma_1\colon(w,x,y,z)\mapsto(-w,x,-y,z),$$

$$\sigma_2\colon(w,x,y,z)\mapsto(w,x,y,\zeta_4z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{z=0\}$	$C_4=\langle\sigma_2\rangle$	G	1	1	yes

We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/4\mathbb{Z})^2.$$

2.44.1 The model is given by

$$X=\{w^2=x^4+y^4+z^4\}\subset\mathbb{P}(2,1,1,1)_{w,x,y,z},\quad G=\langle\sigma_1,\sigma_2\rangle=C_4^2,$$

$$\sigma_1\colon(w,x,y,z)\mapsto(w,x,\zeta_4 y,z),\quad \sigma_2\colon(w,x,y,z)\mapsto(w,x,y,\zeta_4 z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{x=0\}$	$C_2=\langle\sigma_1^2\sigma_2^2\rangle$	G	1	0	
2	$\{y=0\}$	$C_4=\langle\sigma_1\rangle$	G	1	0	yes
3	$\{z=0\}$	$C_4=\langle\sigma_2\rangle$	G	1	0	

Let p_{ij} be the number of intersection points of the images of ξ_i and ξ_j in $X^{(1)}/G$. We record

$$p_{12}=p_{13}=1,\quad p_{23}=2.$$

Since $\xi_1\cap\xi_2\cap\xi_3$ is empty, we have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})\oplus(\mathbb{Z}/4\mathbb{Z}).$$

2.44.2 The model is given by

$$X=\{w^2=x^4+y^4+z^4\}\subset\mathbb{P}(2,1,1,1)_{w,x,y,z},\quad G=\langle\sigma_1,\sigma_2\rangle=C_4^2,$$

$$\sigma_1\colon(w,x,y,z)\mapsto(-w,x,\zeta_4 y,z),\quad \sigma_2\colon(w,x,y,z)\mapsto(w,x,y,\zeta_4 z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{y=0\}$	$C_2=\langle\sigma_1^2\rangle$	G	1	0	
2	$\{x=0\}$	$C_4=\langle\sigma_1\sigma_2\rangle$	G	1	0	yes
3	$\{z=0\}$	$C_4=\langle\sigma_2\rangle$	G	1	0	

Let p_{ij} be the number of intersection points of the images of ξ_i and ξ_j in $X^{(1)}/G$. We record

$$p_{12}=p_{13}=p_{23}=1.$$

Since $\xi_1\cap\xi_2\cap\xi_3$ is empty, we have

$$\mathrm{Br}([X/G]) = \mathbb{Z}/2\mathbb{Z}.$$

• **Automorphisms of Del Pezzo surfaces of degree** 1

1.B2.1 The model is given by

$$X = \{w^2 = z^3 + zL_2(x^2, y^2) + L_3(x^2, y^2)\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z},$$

$$G = C_2^2, \quad \sigma_1\colon (w,x,y,z) \mapsto (-w,x,y,z), \; \sigma_2\colon (w,x,y,z) \mapsto (w,x,-y,z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2 = \langle\sigma_1\rangle$	G	4	1	yes
2	$\{y=0\}$	$C_2 = \langle\sigma_2\rangle$	G	1	0	

The images of ξ_1 and ξ_2 in $X^{(1)}/G$ intersect in three points. We have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^4.$$

1.$\sigma\rho$2.1 The model is given by

$$X = \{w^2 = z^3 + L_3(x^2, y^2)\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z}, \quad G = C_6 \times C_2,$$

$$\sigma_1\colon (w,x,y,z) \mapsto (-w,x,y,\zeta_3 z), \quad \sigma_2\colon (w,x,y,z) \mapsto (w,x,-y,z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2 = \langle\sigma_1^3\rangle$	G	4	0	yes
2	$\{y=0\}$	$C_2 = \langle\sigma_2\rangle$	G	1	0	
3	$\{z=0\}$	$C_3 = \langle\sigma_1^2\rangle$	G	2	0	

The images of ξ_1 and ξ_2 in $X^{(1)}/G$ intersect in one point. We have

$$\mathrm{Br}([X/G]) = 0.$$

1.$\sigma\rho$3 The model is given by

$$X = \{w^2 = z^3 + L_2(x^3, y^3)\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z}, \quad G = C_6 \times C_3,$$

$$\sigma_1\colon (w,x,y,z) \mapsto (-w,x,y,\zeta_3 z), \quad \sigma_2\colon (w,x,y,z) \mapsto (w,x,\zeta_3 y,z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2 = \langle\sigma_1^3\rangle$	G	4	0	yes
2	$\{y=0\}$	$C_3 = \langle\sigma_2\rangle$	G	1	0	
3	$\{x=0\}$	$C_3 = \langle\sigma_1^2\sigma_2\rangle$	G	1	0	
4	$\{z=0\}$	$C_3 = \langle\sigma_1^2\rangle$	G	2	0	

Let p_{ij} be the number of intersection points of the images of ξ_i and ξ_j in $X^{(1)}/G$. We record

$$p_{23} = p_{24} = p_{34} = 1.$$

Since $\xi_2 \cap \xi_3 \cap \xi_4$ is empty, we have

$$\mathrm{Br}([X/G]) = \mathbb{Z}/3\mathbb{Z}.$$

1.ρ3 The model is given by

$$X = \{w^2 = z^3 + L_2(x^3, y^3)\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z}, \quad G = C_3^2,$$

$$\sigma_1\colon (w,x,y,z) \mapsto (w,x,y,\zeta_3 z), \quad \sigma_2\colon (w,x,y,z) \mapsto (w,x,\zeta_3 y,z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{z=0\}$	$C_3 = \langle\sigma_1\rangle$	G	2	0	
2	$\{y=0\}$	$C_3 = \langle\sigma_2\rangle$	G	1	0	yes
3	$\{x=0\}$	$C_3 = \langle\sigma_1^2\sigma_2\rangle$	G	1	0	

Let p_{ij} be the number of intersection points of the images of ξ_i and ξ_j in $X^{(1)}/G$. We record

$$p_{12} = p_{13} = 2, \quad p_{23} = 1.$$

Since $\xi_1 \cap \xi_2 \cap \xi_3$ is empty, we have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/3\mathbb{Z})^3.$$

1.B4.1 The model is given by

$$X = \{w^2 = z^3 + zL_1(x^4, y^4) + x^2 L_1'(x^4, y^4)\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z},$$

$$G = C_2 \times C_4,$$

$$\sigma_1\colon (w,x,y,z) \mapsto (-w,x,y,z), \quad \sigma_2\colon (w,x,y,z) \mapsto (w,x,\zeta_4 y,z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2 = \langle\sigma_1\rangle$	G	4	0	
2	$\{x=0\}$	$C_2 = \langle\sigma_1\sigma_2^2\rangle$	G	1	0	yes
3	$\{y=0\}$	$C_4 = \langle\sigma_2\rangle$	G	1	0	

Let p_{ij} be the number of intersection points of the images of ξ_i and ξ_j in $X^{(1)}/G$. We record

$$p_{12} = 2, \quad p_{13} = 3, \quad p_{23} = 1.$$

Since $\xi_1 \cap \xi_2 \cap \xi_3$ is empty, we have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^4.$$

1.B6.1 The model is given by

$$X = \{w^2 = z^3 + \lambda z x^4 + \mu x^6 + y^6\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z}, \quad G = C_2 \times C_6,$$

$$\sigma_1 \colon (w,x,y,z) \mapsto (-w,x,y,z), \quad \sigma_2 \colon (w,x,y,z) \mapsto (w,x,-\zeta_3 y,z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2 = \langle \sigma_1 \rangle$	G	4	0	
2	$\{x=0\}$	$C_2 = \langle \sigma_1 \sigma_2^3 \rangle$	G	1	0	yes
3	$\{y=0\}$	$C_6 = \langle \sigma_2 \rangle$	G	1	0	

Let p_{ij} be the number of intersection points of the images of ξ_i and ξ_j in $X^{(1)}/G$. We record

$$p_{13} = 3, \quad p_{12} = p_{23} = 1.$$

Since $\xi_1 \cap \xi_2 \cap \xi_3$ is empty, we have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^3.$$

1.$\sigma\rho$6 The model is given by

$$X = \{w^2 = z^3 + x^6 + y^6\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z}, \quad G = C_6^2,$$

$$\sigma_1 \colon (w,x,y,z) \mapsto (-w,x,y,\zeta_3 z), \quad \sigma_2 \colon (w,x,y,z) \mapsto (w,x,-\zeta_3 y,z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2 = \langle \sigma_1^3 \rangle$	G	4	0	
2	$\{z=0\}$	$C_3 = \langle \sigma_1^2 \rangle$	G	2	0	yes
3	$\{x=0\}$	$C_6 = \langle \sigma_1^5 \sigma_2 \rangle$	G	1	0	
4	$\{y=0\}$	$C_6 = \langle \sigma_2 \rangle$	G	1	0	

Let p_{ij} be the number of intersection points of the images of ξ_i and ξ_j in $X^{(1)}/G$. We record

$$p_{14} = 3, \quad p_{13} = p_{23} = p_{24} = p_{34} = 1.$$

Since the intersection of any three of the four curves ξ_i, $i = 1,2,3,4$ is empty, we have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^3 \oplus (\mathbb{Z}/3\mathbb{Z}).$$

1.ρ6 The model is given by

$$X = \{w^2 = z^3 + x^6 + y^6\} \subset \mathbb{P}(3,1,1,2)_{w,x,y,z}, \quad G = C_3 \times C_6,$$

$$\sigma_1 \colon (w,x,y,z) \mapsto (w,x,y,\zeta_3 z), \quad \sigma_2 \colon (w,x,y,z) \mapsto (w,x,-\zeta_3 y,z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{z=0\}$	$C_3=\langle\sigma_1\rangle$	G	2	0	
2	$\{x=0\}$	$C_3=\langle\sigma_1\sigma_2^2\rangle$	G	1	0	yes
3	$\{y=0\}$	$C_6=\langle\sigma_2\rangle$	G	1	0	

Let p_{ij} be the number of intersection points of the images of ξ_i and ξ_j in $X^{(1)}/G$. We record

$$p_{13}=2, \quad p_{12}=p_{23}=1.$$

Since $\xi_1\cap\xi_2\cap\xi_3$ is empty, we have

$$\mathrm{Br}([X/G])=(\mathbb{Z}/3\mathbb{Z})^2.$$

1.B6.2 The model is given by

$$X=\{w^2=z^3+\lambda zx^2y^2+x^6+y^6\}\subset\mathbb{P}(3,1,1,2)_{w,x,y,z}, \quad G=C_2\times C_6,$$

$$\sigma_1\colon (w,x,y,z)\mapsto(-w,x,y,z), \quad \sigma_2\colon (w,x,y,z)\mapsto(w,x,-\zeta_3y,\zeta_3z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma_1\rangle$	G	4	1	
2	$\{x=0\}$	$C_2=\langle\sigma_1\sigma_2^3\rangle$	G	1	0	yes
3	$\{y=0\}$	$C_2=\langle\sigma_2^3\rangle$	G	1	0	

Let p_{ij} be the number of intersection points of the images of ξ_i and ξ_j in $X^{(1)}/G$. We record

$$p_{12}=p_{13}=p_{23}=1.$$

Since $\xi_1\cap\xi_2\cap\xi_3$ is empty, we have

$$\mathrm{Br}([X/G])=(\mathbb{Z}/2\mathbb{Z})^3.$$

1.B12 The model is given by

$$X=\{w^2=z^3+\lambda zx^4+y^6\}\subset\mathbb{P}(3,1,1,2)_{w,x,y,z}, \quad G=C_2\times C_{12},$$

$$\sigma_1\colon (w,x,y,z)\mapsto(-w,x,y,z), \quad \sigma_2\colon (w,x,y,z)\mapsto(\zeta_4w,x,\zeta_{12}y,-z).$$

The fixed curves stratification is

i	Curves ξ_i	I_{ξ_i}	D_{ξ_i}	$\mathrm{g}(\xi_i)$	$\mathrm{g}(\xi_i/D_{\xi_i})$	Standard form
1	$\{w=0\}$	$C_2=\langle\sigma_1\rangle$	G	4	0	
2	$\{x=0\}$	$C_4=\langle\sigma_2^3\rangle$	G	1	0	yes
3	$\{y=0\}$	$C_6=\langle\sigma_1\sigma_2^2\rangle$	G	1	0	

Let p_{ij} be the number of intersection points of the images of ξ_i and ξ_j in $X^{(1)}/G$. We record

$$p_{13} = 2, \quad p_{12} = p_{23} = 1.$$

Since $\xi_1 \cap \xi_2 \cap \xi_3$ is empty, we have

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^2.$$

4.3. **Tables.** We record the above computations in the following tables.

Cyclic groups G

Label in [1]	Group G	Surface X	$\mathrm{Br}([X/G])$
0.n	C_n	$\mathbb{P}^2$	0
C.2	C_2	conic bundles	vary
2.G	C_2	dP_2	$(\mathbb{Z}/2\mathbb{Z})^6$
1.B	C_2	dP_1	$(\mathbb{Z}/2\mathbb{Z})^8$
C.ro.m	C_{2m}	conic bundles	vary
C.re.m	C_{2m}	conic bundles	vary
3.3	C_3	dP_3	$(\mathbb{Z}/3\mathbb{Z})^2$
1.ρ	C_3	dP_1	$(\mathbb{Z}/3\mathbb{Z})^4$
2.4	C_4	dP_2	$(\mathbb{Z}/4\mathbb{Z})^2$
1.B2.2	C_4	dP_1	$(\mathbb{Z}/2\mathbb{Z})^4$
1.5	C_5	dP_1	$(\mathbb{Z}/5\mathbb{Z})^2$
3.6.1	C_6	dP_3	0
3.6.2	C_6	dP_3	$(\mathbb{Z}/2\mathbb{Z})^2$
2.G3.1	C_6	dP_2	0
2.G3.2	C_6	dP_2	$(\mathbb{Z}/2\mathbb{Z})^2$
2.6	C_6	dP_2	$(\mathbb{Z}/3\mathbb{Z})^2$
1.$\sigma\rho$	C_6	dP_1	0
1.ρ2	C_6	dP_1	0
1.B3.1	C_6	dP_1	0
1.B3.2	C_6	dP_1	$(\mathbb{Z}/2\mathbb{Z})^4$
1.6	C_6	dP_1	$(\mathbb{Z}/6\mathbb{Z})^2$
1.B4.2	C_8	dP_1	$\mathbb{Z}/2\mathbb{Z}$
3.9	C_9	dP_3	0
1.B5	C_{10}	dP_1	0
3.12	C_{12}	dP_3	0
2.12	C_{12}	dP_2	0

1.$\sigma\rho$2.2	C_{12}	dP_1	0
2.G7	C_{14}	dP_2	0
1.ρ5	C_{15}	dP_1	0
2.G9	C_{18}	dP_2	0
1.B10	C_{20}	dP_1	0
1.$\sigma\rho$4	C_{24}	dP_1	0
1.$\sigma\rho$5	C_{30}	dP_1	0

Noncyclic groups G

Label in [1]	Group G	Surface X	$\mathrm{Br}([X/G])$
0.mn	$C_n \times C_m$	$\mathbb{P}^1 \times \mathbb{P}^1$	$\mathbb{Z}/\gcd(m,n)\mathbb{Z}$
P1.22n	$C_2 \times C_{2n}$	$\mathbb{P}^1 \times \mathbb{P}^1$	0
P1.222n	$C_2^2 \times C_{2n}$	$\mathbb{P}^1 \times \mathbb{P}^1$	$(\mathbb{Z}/2\mathbb{Z})^2$
P1.22.1	C_2^2	$\mathbb{P}^1 \times \mathbb{P}^1$	0
P1.222	C_2^3	$\mathbb{P}^1 \times \mathbb{P}^1$	$(\mathbb{Z}/2\mathbb{Z})^2$
P1.2222	C_2^4	$\mathbb{P}^1 \times \mathbb{P}^1$	$(\mathbb{Z}/2\mathbb{Z})^4$
P1s.24	$C_2 \times C_4$	$\mathbb{P}^1 \times \mathbb{P}^1$	0
P1s.222	C_2^3	$\mathbb{P}^1 \times \mathbb{P}^1$	$(\mathbb{Z}/2\mathbb{Z})^3$
0.V9	C_3^2	$\mathbb{P}^2$	0
4.222	C_2^3	dP_4	$(\mathbb{Z}/2\mathbb{Z})^4$
4.2222	C_2^4	dP_4	$(\mathbb{Z}/2\mathbb{Z})^6$
4.42	$C_4 \times C_2$	dP_4	$(\mathbb{Z}/2\mathbb{Z})^2$
3.33.1	C_3^2	dP_3	$(\mathbb{Z}/3\mathbb{Z})^2$
3.33.2	C_3^2	dP_3	$(\mathbb{Z}/3\mathbb{Z})^2$
3.36	$C_3 \times C_6$	dP_3	$\mathbb{Z}/3\mathbb{Z}$
3.333	C_3^3	dP_3	$(\mathbb{Z}/3\mathbb{Z})^3$
2.G2	C_2^2	dP_2	$(\mathbb{Z}/2\mathbb{Z})^5$
2.G4.1	$C_2 \times C_4$	dP_2	$(\mathbb{Z}/2\mathbb{Z})^3$
2.G4.2	$C_2 \times C_4$	dP_2	$(\mathbb{Z}/2\mathbb{Z})^3$
2.G6	$C_2 \times C_6$	dP_2	$\mathbb{Z}/2\mathbb{Z}$
2.G8	$C_2 \times C_8$	dP_2	$(\mathbb{Z}/2\mathbb{Z})^2$
2.G12	$C_2 \times C_{12}$	dP_2	$\mathbb{Z}/2\mathbb{Z}$
2.G22	C_2^3	dP_2	$(\mathbb{Z}/2\mathbb{Z})^6$
2.G24	$C_2^2 \times C_4$	dP_2	$(\mathbb{Z}/2\mathbb{Z})^4$
2.G44	$C_2 \times C_4^2$	dP_2	$(\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z})$

2.24.1	$C_2 \times C_4$	dP_2	$(\mathbb{Z}/2\mathbb{Z})^3$
2.24.2	$C_2 \times C_4$	dP_2	$(\mathbb{Z}/4\mathbb{Z})^2$
2.44.1	C_4^2	dP_2	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$
2.44.2	C_4^2	dP_2	$\mathbb{Z}/2\mathbb{Z}$
1.B2.1	C_2^2	dP_1	$(\mathbb{Z}/2\mathbb{Z})^4$
1.$\sigma\rho$2.1	$C_6 \times C_2$	dP_1	0
1.$\sigma\rho$3	$C_6 \times C_3$	dP_1	$\mathbb{Z}/3\mathbb{Z}$
1.ρ3	C_3^2	dP_1	$(\mathbb{Z}/3\mathbb{Z})^3$
1.B4.1	$C_2 \times C_4$	dP_1	$(\mathbb{Z}/2\mathbb{Z})^4$
1.B6.1	$C_2 \times C_6$	dP_1	$(\mathbb{Z}/2\mathbb{Z})^3$
1.$\sigma\rho$6	C_6^2	dP_1	$(\mathbb{Z}/2\mathbb{Z})^3 \oplus (\mathbb{Z}/3\mathbb{Z})$
1.ρ6	$C_3 \times C_6$	dP_1	$(\mathbb{Z}/3\mathbb{Z})^2$
1.B6.2	$C_2 \times C_6$	dP_1	$(\mathbb{Z}/2\mathbb{Z})^3$
1.B12	$C_2 \times C_{12}$	dP_1	$(\mathbb{Z}/2\mathbb{Z})^2$

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