EQUIVARIANT GEOMETRY OF SINGULAR CUBIC THREEFOLDS

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Throughout, we work with algebraically closed field k of characteristic 0.

Among the central problems in birational geometry is the *linearizability* problem, aimed at determining whether or not a regular action from a finite group on a rational variety is equivariantly birational to a regular action on projective spaces. Motivations to study this problem arise from both geometry and arithmetic. It is part of the long-standing program of identifying conjugacy classes of the Cremona group, the group of birational automorphisms of $\mathbb{P}^n$, which remains largely open in dimension 3 or higher; it is also closely related to the *rationality problem* over nonclosed fields, where the Galois action is considered an analogue of the group action.

In [CTZ24] and [CMTZ24], we study the linearizability problem of singular cubic threefolds. Precisely, we are interested in the following:

Main problem: Let X be a cubic hypersurface in $\mathbb{P}^4(k)$ with ADE singularities and $G \subset \mathrm{Aut}(X)$. When is there a G-equivariant birational map $X \dashrightarrow \mathbb{P}^3$ where the G-action on $\mathbb{P}^3$ is regular and generically free? If such a birational map exists, we call the G-action on X is *linearizable*; we also say it is *stably linearizable* if the G-action on $X \times \mathbb{P}^r$ is linearizable for some $r \in \mathbb{N}$ and trivial action on the $\mathbb{P}^r$ factor.

From now on, let X denote such a singular cubic threefold. Beginning with the recent classification of configurations of isolated singularities on cubic threefolds [Vik23], we proceed to identify the automorphism groups $\mathrm{Aut}(X)$ in most cases, followed by a case-by-case analysis of linearizability. These cubic threefolds are rational via projection from a singular point. The rich geometry behind them allows us to explore

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and showcase the applicability of a wide range of tools. In particular, among the linearization constructions we use (when they exist) are

- Projection from a G-fixed singular point.
- Unprojection from a G-invariant plane to an intersection of two quadrics in $\mathbb{P}^5$, followed by projection from a G-invariant line.
- Equivariant birational map to a quadric threefold, followed by projection from a G-fixed point there.

When in absence of an obvious linearization map, we look for various obstructions to the linearizability, including:

Intermediate Jacobian. Let $k = \mathbb{C}$. The seminal work by Clemens and Griffiths proves that smooth cubic threefolds are irrational over $\mathbb{C}$ by showing their intermediate Jacobians are not isomorphic to the Jacobian of a (possibly reducible) curve as principally polarized abelian varieties. Since singular cubic threefolds are rational, their intermediate Jacobians are indeed Jacobians of curves. But this may fail equivariantly – the group action on the intermediate Jacobian may not come from the action on the curve when the curve is non-hyperelliptic:

Proposition 1. *Let X be a singular cubic threefold with $2\mathsf{A}_1$ or $2\mathsf{A}_2$-singularities over $\mathbb{C}$, and $G \subset \mathrm{Aut}(X)$. Then the G-action on X is linearizable if and only if it fixes two singular points.*

Sketch proof. In these cases, the intermediate Jacobian $\mathrm{IJ}(X)$ is isomorphic to the Jacobian of a smooth plane quartic curve C. Then $\mathrm{Aut}(\mathrm{J}(C)) = \mathrm{Aut}(C) \times C_2$. When G switches two singular points, the G-actions on $\mathrm{IJ}(X)$ and $\mathrm{J}(C)$ differ by the C_2-factor. So the G-action on X does not come from blowups of curves in $\mathbb{P}^3$ and we conclude it is not linearizable by the equivariant weak factorization theorem. $\square$

Cohomology. Given a G-action on X, there is a natural G-lattice $\mathrm{Pic}(\tilde{X})$, where $\tilde{X}$ is an equivariant desingularization of X. When performing equivariant blowups, we change the Picard lattice by adding a copy of $\mathbb{Z}^r$ to it, with some $r \in \mathbb{N}$ and the G-action permuting factors of $\mathbb{Z}^r$. Therefore, if the G-action on X is stably linearizable, $\mathrm{Pic}(\tilde{X})$ is a *stably permutation module*, i.e., $\mathrm{Pic}(\tilde{X}) \oplus M$ is a permutation module for some permutation module M. In particular, this implies the vanishing of the first group cohomology

$$\mathrm{H}^1(H, \mathrm{Pic}(\tilde{X})) = 0, \quad \forall H \subset G. \tag{1}$$

We call the failure of (1) the **(H1)**-obstruction to stable linearizability. This can be effectively checked given a presentation of the Picard lattice. We compute (**H1**)-obstruction in each configuration of singularities on a cubic threefold.

Proposition 2. *Let X be a cubic threefold with isolated singularities, and $\tilde{X} \to X$ an* $\mathrm{Aut}(X)$*-equivariant resolution of singularities. Then*

- $\mathrm{Pic}(\tilde{X})$ *is a permutation module for* $\mathrm{Aut}(X)$ *if X is not of one of the following configurations of singularities*
$$\mathsf{6A}_1 \textit{ in linear general position }, \quad \mathsf{8A}_1, \quad \mathsf{9A}_1, \quad \mathsf{10A}_1,$$
$$\mathsf{2A}_5, \quad \mathsf{2D}_4 + \mathsf{2A}_1 \quad \textit{and} \quad \mathsf{3D}_4.$$
- *For each of the cubic threefolds X with singularities in the list above, if the* $\mathrm{Aut}(X)$*-action does not fix any singular point, then it has an* (**H1**)*-obstruction.*

Specialization. There is an extensive literature on the singularity theory and the moduli space of cubic threefolds, facilitating a natural application of the specialization/degeneration technique. It is classically known that, locally, ADE singularities T specializes to T' if and only if the Dynkin diagram of the root system of T is an *induced subgraph* of that of T'. For cubic threefolds, this result extends globally [Vik23, Remark 1.19]. In presence of the group action, we apply the Kresch-Tschinkel equivariant specialization theorem:

Theorem 3 ([KT22, Corollary 6.8],[CTZ24, Proposition 2.9])**.** *Let k be an uncountable algebraically closed field of characteristic zero and G a finite group. Let $\pi : \mathcal{X} \to B$ be a G-equivariant flat and projective morphism onto a smooth curve over k, such that*

- *G acts trivially on B and generically freely on the fibers of π,*
- *for some $b_0 \in B$, the special fiber $\mathcal{X}_0$ is irreducible, has so called BG-rational singularities, and the G-action on $\mathcal{X}_0$ is not linearizable.*

Then, for very general $b \in B$, the G-action on the special fiber $\mathcal{X}_b$ is not linearizable.

The specialization technique allows to exhibit "invisible" linearizability obstruction when applied to the family where the general fibers do not carry obstructions themselves but the central fiber does:

Example 4. Let $\mathcal{X} \to \mathbb{A}^1_k$ be a family of cubic threefolds given by

$$x_1x_2x_3 + (x_1 + x_2 + x_3)x_4x_5 + (x_4 + x_5)(x_4 + bx_5)(bx_4 + x_5) = 0$$

with parameter $b \in k$. Consider the $G = C_3$ action generated by permuting coordinates x_1, x_2, x_3. For a very general $b \in k$, the fiber is a cubic threefold with $3\mathsf{A}_1$-singularities; the central fiber above $b = 0$ is a cubic threefold X_0 with $9\mathsf{A}_1$-singularities. The G-action on X_0 has an **(H1)**-obstruction. So a very general member in $\mathcal{X}$ is not G-stably linearizable while there is no **(H1)**-obstruction to itself.

Burnside Formalism. The Burnside formalism is a powerful tool to study equvariant birational geometry recently developed by Kontsevich, Kresch, Pestun, and Tschinkel [KPT23], [KT22]. To an appropriate model X of dimension n with a G-action, it assigns a *class* $[X \circlearrowleft G]$ taking values in the *Burnside group* $\mathrm{Burn}_n(G)$, an abelian group defined with *symbols* as generators and certain *blowup* relations ensuring the class $[X \circlearrowleft G]$ is invariant under equivariant blowups. The class captures information of all strata with nontrivial stabilizers and the residue G-action on them, complementing many classical framework e.g., birational rigidity. It can be computationally challenging to apply the Burnside formalism in some cases. We illustrate a simple application in the following example, using *incompressible symbols*.

Example 5. Consider the cubic threefold X with $4\mathsf{A}_2$-singularities given by

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + x_5^3 = 0 \tag{2}$$

and the $G = C_2^2$ action generated by τ switching coordinates $x_1 \leftrightarrow x_2$ and $\iota : x_3 \leftrightarrow x_4$. Then τ fixes a smooth cubic surface $S = \{x_1 = x_2\} \cap X$ with a residue ι-action. So the class $[X \circlearrowleft G]$ contains a symbol

$$(\langle\tau\rangle, \langle\iota\rangle \circlearrowright k(S), \beta).$$

Since ι fixes a genus 1 curve on S, we know $\mathrm{H}^1(\langle\iota\rangle, \mathrm{Pic}(S)) = (\mathbb{Z}/2)^2$. Then S is not ι-equivariantly birational to the exceptional divisor of any blowup of a *standard model* in Burnside formalism. This implies that the symbol (2) is *incompressible*, meaning it is a free generator of $\mathrm{Burn}_3(G)$. On the other hand, the class $[\mathbb{P}^3 \circlearrowleft G]$ of any linear action does not contain the symbol (2). So the class $[X \circlearrowleft G]$ is different from that of any linear action, and we conclude that the G-action on X is not linearizable. Note that none of other obstructions mentioned above is applicable to this case.

References

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